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IOANNIS BOLYAI DE BOLYA

APPENDIX

SCIENTIAM SPATII ABSOLUTE VERAM EXHIBENS.

IOANNIS BOLYAI DE BOLYA

APPENDIX

SCIENTIAM SPATII ABSOLUTE VERAM EXHIBENS: A VERITATE AUT FALSITATE AXIOMATIS XI. EUCLIDEI, A PRIORI HAUD UNQUAM DECIDENDA, INDEPENDENTEM: ADIECTA AD CASUM FALSITATIS QUADRATURA CIRCULI GEOMETRICA.

EDITIO NOVA

OBLATA AB ACADEMIA SCIENTIARUM HUNGARICA
AD DIEM NATALEM CENTESIMUM AUCTORIS CONCELEBRANDUM.

EDIDERUNT

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APPENDIX.

SCIENTIAM SPATII *absolute veram* exhibens:

a veritate aut falsitate Axiomatis XI *Euclidei* (*a priori haud unquam decidenda*) *independentem*; adjecta ad casum falsitatis, quadratura circuli geometrica.

Auctore JOHANNE BOLYAI de eadem, Geometrarum in Exercitu Caesareo Regio Austriaco Castrensium Capitaneo.

EXPLICATIO SIGNORUM.

$\widetilde{\mathfrak{ab}}$ denotet complexum *omnium* punctorum cum punctis $\mathfrak{a}$, $\mathfrak{b}$ in recta sitorum.

$\mathfrak{a}\tilde{\mathfrak{b}}$ « rectæ $\widetilde{\mathfrak{ab}}$ in $\mathfrak{a}$ bifariam sectæ dimidium illud, quod punctum $\mathfrak{b}$ complectitur.

$\overline{\mathfrak{abc}}$ « complexum *omnium* punctorum, quæ cum punctis $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ (non in eadem recta sitis) in eodem plano sunt.

$\mathfrak{ab}\bar{\mathfrak{c}}$ « plani $\overline{\mathfrak{abc}}$ per $\widetilde{\mathfrak{ab}}$ bifariam secti dimidium, punctum $\mathfrak{c}$ complectens.

$\mathfrak{abc}$ « portionum, in quas $\overline{\mathfrak{abc}}$ per complexum rectarum $\mathfrak{b}\tilde{\mathfrak{a}}$, $\mathfrak{b}\bar{\mathfrak{c}}$ dividitur, *minorem*; sive *angulum*, cuius $\mathfrak{b}\tilde{\mathfrak{a}}$, $\mathfrak{b}\bar{\mathfrak{c}}$ crura sunt.

$\mathfrak{abcd}$ « (si $\mathfrak{d}$ in $\mathfrak{abc}$ sit et $\widetilde{\mathfrak{ba}}$, $\widetilde{\mathfrak{cd}}$ se invicem non secent) portionem ipsius $\mathfrak{abc}$ inter $\mathfrak{b}\tilde{\mathfrak{a}}$, $\mathfrak{bc}$, $\mathfrak{c}\bar{\mathfrak{d}}$ comprehensam; $\mathfrak{bacd}$ vero portionem plani $\overline{\mathfrak{abc}}$ inter $\widetilde{\mathfrak{ab}}$, $\widetilde{\mathfrak{cd}}$ sitam.

R « angulum rectum.

$\mathfrak{ab} \bumpeq \mathfrak{cd}$ « $\mathfrak{cab} = \mathfrak{acd}$.

$\equiv$ « congruens.*

$x \leadsto a$ « x tendere ad limitem a.

$\bigcirc r$ « peripheriam circuli radii r.

$\odot r$ « aream circuli radii r.

* Sit fas, signo hocce, quo summus Geometra *GAUSS numeros congruos* insignivit, congruentiam geometricam quoque denotare: nulla ambiguitate exinde metuenda.

§. 1.

Si rectam $a\tilde{m}$ non secet plani eiusdem recta $b\tilde{n}$, at secet quævis $b\tilde{p}$ (in abn): designetur hoc per Fig. 1.

$$bn \,|||\, am.$$

Dari talem $b\tilde{n}$, et quidem *unicam*, e quovis puncto b (extra $\overline{am}$), atque

$$bam + abn \text{ non } > 2R$$

esse patet; nam bc circa b mota, donec

$$bam + abc = 2R$$

fiat, $b\tilde{c}$ aliquando *primo* non secat $a\tilde{m}$, estque tunc $bc \,|||\, am$.

Nec non patet esse $bn \,|||\, em$, ubivis sit e in $\overline{am}$ (supponendo in omnibus talibus casibus esse $am > ae$).

Et si, puncto c in $a\tilde{m}$ abeunte in infinitum, semper sit $cd = cb$: erit semper

$$cdb = (cbd < nbc);$$

ast $nbc \to 0$; adeoque et $adb \to 0$.

§. 2.

Si $bn \,|||\, am$; *est quoque* $cn \,|||\, am$. Fig. 2.

Nam sit d ubicunque in $macn$. Si c in $b\tilde{n}$ sit; $b\tilde{d}$ secat $a\tilde{m}$ (propter $bn \,|||\, am$), adeoque et $c\tilde{d}$ secat $a\tilde{m}$; si vero c in $b\tilde{p}$ fuerit; sit $bq \,|||\, cd$: cadit $b\tilde{q}$ in abn (§. 1.) secatque $a\tilde{m}$, adeoque et $c\tilde{d}$ secat $a\tilde{m}$. Quævis $c\tilde{d}$ igitur (in acn) secat in utroque casu $a\tilde{m}$ absque eo, ut $c\tilde{n}$ ipsam $a\tilde{m}$ secet. Est ergo semper $cn \,|||\, am$.

§. 3.

Fig. 2. *Si tam* $\mathfrak{br}$ *quam* $\mathfrak{cs}$ *sit* $|||\,\mathfrak{am}$, *et* $\mathfrak{c}$ *non sit in* $\widetilde{\mathfrak{br}}$; *tum* $\mathfrak{b}\tilde{\mathfrak{r}}$, $\mathfrak{c}\tilde{\mathfrak{s}}$ *se invicem haud secant.*

Si enim $\mathfrak{b}\tilde{\mathfrak{r}}$, $\mathfrak{c}\tilde{\mathfrak{s}}$ punctum $\mathfrak{d}$ commune haberent; (per §. 2.) essent $\mathfrak{dr}$ et $\mathfrak{ds}$ simul $|||\,\mathfrak{am}$, caderetque (§. 1.) $\mathfrak{d}\tilde{\mathfrak{s}}$ in $\mathfrak{d}\tilde{\mathfrak{r}}$ et $\mathfrak{c}$ in $\widetilde{\mathfrak{br}}$ (contra hyp.).

§. 4.

Fig. 3. *Si* $\mathfrak{man} > \mathfrak{mab}$; *pro quovis puncto* $\mathfrak{b}$ *ipsius* $\mathfrak{a}\tilde{\mathfrak{b}}$ *datur tale* $\mathfrak{c}$ *in* $\mathfrak{a}\tilde{\mathfrak{m}}$, *ut sit* $\mathfrak{bcm} = \mathfrak{nam}$.

Nam datur (per §. 1.) $\mathfrak{bdm} > \mathfrak{nam}$, adeoque $\mathfrak{mdp} = \mathfrak{man}$, caditque $\mathfrak{b}$ in $\mathfrak{nadp}$. Si igitur $\mathfrak{nam}$ iuxta $\mathfrak{am}$ feratur, usquequo $\mathfrak{a}\tilde{\mathfrak{n}}$ in $\mathfrak{d}\tilde{\mathfrak{p}}$ veniat; aliquando $\mathfrak{a}\tilde{\mathfrak{n}}$ per $\mathfrak{b}$ transiisse, et aliquod $\mathfrak{bcm} = \mathfrak{nam}$ esse oportet.

§. 5.

Fig. 1. *Si* $\mathfrak{bn}\,|||\,\mathfrak{am}$, *datur tale punctum* $\mathfrak{f}$ *in* $\widetilde{\mathfrak{am}}$, *ut sit* $\mathfrak{fm} \bumpeq \mathfrak{bn}$.

Nam (per §. 1.) datur $\mathfrak{bcm} > \mathfrak{cbn}$; et si $\mathfrak{ce} = \mathfrak{cb}$, adeoque $\mathfrak{ec} \bumpeq \mathfrak{bc}$; patet esse $\mathfrak{bem} < \mathfrak{ebn}$. Feratur $\mathfrak{p}$ per $\mathfrak{ec}$, angulo $\mathfrak{bpm}$ semper u, et angulo $\mathfrak{pbn}$ semper v dicto; patet u esse prius ei simultaneo v minus, posterius vero esse maius. Crescit vero u a $\mathfrak{bem}$ usque $\mathfrak{bcm}$ *continuo*; cum (per §. 4.) *nullus* angulus $> \mathfrak{bem}$ et $< \mathfrak{bcm}$ detur, cui u aliquando $=$ non fiat; pariter decrescit v ab $\mathfrak{ebn}$ usque $\mathfrak{cbn}$ continuo: datur itaque in $\mathfrak{ec}$ tale $\mathfrak{f}$, ut $\mathfrak{bfm} = \mathfrak{fbn}$ sit.

§. 6.

Si $\mathfrak{bn}\,|||\,\mathfrak{am}$, *atque ubivis sit* $\mathfrak{e}$ *in* $\widetilde{\mathfrak{am}}$ *et* $\mathfrak{g}$ *in* $\widetilde{\mathfrak{bn}}$: *tum* $\mathfrak{gn}\,|||\,\mathfrak{em}$ *et* $\mathfrak{em}\,|||\,\mathfrak{gn}$.

Nam (per §. 1.) est $\mathfrak{bn}\,|||\,\mathfrak{em}$, et hinc (per §. 2.) $\mathfrak{gn}\,|||\,\mathfrak{em}$.

Si porro $\mathfrak{fm} \bumpeq \mathfrak{bn}$ (§. 5.); tum $\mathfrak{mfbn} \bumpeq \mathfrak{nbfm}$, adeoque (cum $\mathfrak{bn}\,|||\,\mathfrak{fm}$ sit) etiam $\mathfrak{fm}\,|||\,\mathfrak{bn}$, et (per præc.) $\mathfrak{em}\,|||\,\mathfrak{gn}$.

§. 7.

Si tam bn *quam* cp *sit* ||| am, *et* c *non sit in* $\overline{bn}$: *est etiam* bn ||| cp. Fig. 4.

Nam $b\overline{n}$, $c\overline{p}$ se invicem non secant (§. 3.); sunt vero am, bn, cp aut in plano, aut non; atque in casu primo am aut in bncp est, aut non.

Si am, bn, cp in plano sint, et am in bncp cadat; tum quævis $b\overline{q}$ (in nbc) secat $\overline{am}$ in aliquo puncto d (quia bn ||| am); porro cum dm ||| cp sit (§. 6.), patet $d\overline{q}$ secare $c\overline{p}$, adeoque esse bn ||| cp.

Si vero bn, cp in eadem plaga ipsius am sint; tum aliqua earum ex. gr. cp *intra* duas reliquas $\overline{bn}$, $\overline{am}$ cadit; quævis $b\overline{q}$ (in nba) autem secat $a\overline{m}$, adeoque et ipsam $\overline{cp}$. Est itaque bn ||| cp.

Si mab, mac *angulum* efficiant: tum cbn cum abn nonnisi $b\overline{n}$, $a\overline{m}$ vero (in abn) cum $b\overline{n}$, adeoque nbc quoque cum $a\overline{m}$, nihil commune habent. Per quamvis $b\overline{d}$ (in nba) autem positum $bc\overline{d}$ secat $a\overline{m}$, quia (propter bn ||| am) $b\overline{d}$ secat $a\overline{m}$. Moto itaque $bc\overline{d}$ circa bc, donec ipsam $a\overline{m}$ *prima vice* deserat, postremo cadet $bc\overline{d}$ in $bc\overline{n}$. Eadem ratione cadet idem in $bc\overline{p}$; cadit igitur bn in bcp. Porro si br ||| cp; tum (quia etiam am ||| cp) pari ratione cadit br in bam; nec non (propter br ||| cp) in bcp. Itaque $b\overline{r}$ ipsis mab, pcb commune, nempe ipsum $b\overline{n}$ est, atque hinc bn ||| cp.

Si igitur cp ||| am, et b extra $\overline{cam}$ sit: tum sectio ipsorum bam, bcp, nempe $b\overline{n}$ est ||| tam ad am, quam ad cp.*

§. 8.

Si bn ||| et ≏ cp (vel brevius bn ||| ≏ cp), *atque* am (*in* nbcp) *rectam* bc *perpendiculariter bisecet*; *tum* bn ||| am. Fig. 5.

Si enim $b\overline{n}$ secaret $a\overline{m}$, etiam $c\overline{p}$ secaret $a\overline{m}$ in eodem puncto (cum mabn ≏ macp), quod et ipsis $b\overline{n}$, $c\overline{p}$ commune esset, quamvis bn ||| cp sit. Quævis $b\overline{q}$ (in cbn) vero secat $c\overline{p}$; adeoque secat $b\overline{q}$ etiam $a\overline{m}$. Consequenter bn ||| am.

* Casu tertio *praemisso* duo priores, adinstar casus secundi §. 10. brevius ac elegantius simul absolvi possunt. (Ed. I. Tom. I. Errata Appendicis).

§. 9.

Fig. 6. *Si* $\mathfrak{bn} ||| \mathfrak{am}$, $\mathfrak{map} \perp \mathfrak{mab}$, *atque angulus, quem* $\mathfrak{nb\delta}$ cum $\mathfrak{nba}$ (in ea plaga ipsius $\mathfrak{mabn}$, ubi $\mathfrak{map}$ est) *facit, sit* $< R$: *tum* $\mathfrak{map}$ et $\mathfrak{nb\delta}$ *se invicem secant.*

Nam sit

$$\mathfrak{bam} = R,\ \mathfrak{ac} \perp \mathfrak{bn}$$

(sive in $\mathfrak{b}$ cadat $\mathfrak{c}$, sive non), et

$$\mathfrak{ce} \perp \mathfrak{bn} \text{ (in } \mathfrak{nb\tilde{\delta}});$$

erit (per hyp.) $\mathfrak{ace} < R$, et $\mathfrak{af}$ ($\perp \mathfrak{ce}$) in $\mathfrak{ace}$ cadet. Sit $\mathfrak{a\tilde{p}}$ sectio (punctum $\mathfrak{a}$ commune habentium) $\mathfrak{ab\tilde{f}}$ et $\mathfrak{am\tilde{p}}$; erit

$$\mathfrak{bap} = \mathfrak{bam} = R$$

(cum sit $\mathfrak{bam} \perp \mathfrak{map}$). Si denique $\mathfrak{ab\tilde{f}}$ in $\mathfrak{ab\tilde{n}}$ ponatur (a et b manentibus); cadet $\mathfrak{a\tilde{p}}$ in $\mathfrak{a\tilde{m}}$; atque cum

$$\mathfrak{ac} \perp \mathfrak{bn} \quad \text{et} \quad \mathfrak{af} < \mathfrak{ac}$$

sit, patet $\mathfrak{af}$ *intra* $\mathfrak{b\tilde{n}}$ terminari, adeoque $\mathfrak{bf}$ in $\mathfrak{abn}$ cadere. Secat autem $\mathfrak{b\tilde{f}}$ ipsam $\mathfrak{a\tilde{p}}$ in *hoc* situ (quia $\mathfrak{bn} ||| \mathfrak{am}$), adeoque etiam in situ *primo* $\mathfrak{a\tilde{p}}$ et $\mathfrak{b\tilde{f}}$ se invicem secant; estque punctum sectionis ipsis $\mathfrak{ma\tilde{p}}$ et $\mathfrak{nb\tilde{\delta}}$ commune: secant itaque $\mathfrak{ma\tilde{p}}$ et $\mathfrak{nb\tilde{\delta}}$ se invicem.

Facile exhinc sequitur $\mathfrak{ma\tilde{p}}$ et $\mathfrak{nb\tilde{\delta}}$ se mutuo secare, si summa internorum, quos cum $\mathfrak{mabn}$ efficiunt, $< 2R$ sit.

§. 10.

Fig. 7. *Si tam* $\mathfrak{bn}$ *quam* $\mathfrak{cp}$ *sit* $||| \bumpeq \mathfrak{am}$; *est etiam* $\mathfrak{bn} ||| \bumpeq \mathfrak{cp}$.

Nam $\mathfrak{mab}$ et $\mathfrak{mac}$ aut *angulum* efficiunt, aut in plano sunt.

Si prius; bisecet $\overset{\frown}{\mathfrak{q\delta f}}$ rectam $\mathfrak{ab}$ perpendiculariter; erit $\mathfrak{\delta q} \perp \mathfrak{ab}$, adeoque $\mathfrak{\delta q} ||| \mathfrak{am}$ (§. 8.); pariter si $\overset{\frown}{\mathfrak{ers}}$ bisecet rectam $\mathfrak{ac}$ perpendiculariter, est $\mathfrak{er} ||| \mathfrak{am}$; unde $\mathfrak{\delta q} ||| \mathfrak{er}$ (§. 7.). Facile hinc (per §. 9.) consequitur, $\overset{\frown}{\mathfrak{q\delta f}}$

et $\overline{\mathfrak{ers}}$ se mutuo secare, et sectionem $\overline{\mathfrak{fs}}$ esse $|||\ \mathfrak{dq}$ (§. 7.), atque (propter $\mathfrak{bn}\ |||\ \mathfrak{dq}$) esse etiam

$$\mathfrak{fs}\ |||\ \mathfrak{bn}.$$

Est porro (pro quovis puncto ipsius $\overline{\mathfrak{fs}}$)

$$\mathfrak{fb} = \mathfrak{fa} = \mathfrak{fc},$$

caditque $\overline{\mathfrak{fs}}$ in planum $\overline{\mathfrak{tgf}}$, rectam $\mathfrak{bc}$ perpendiculariter bisecans. Est vero (per §. 7.) (cum sit $\mathfrak{fs}\ |||\ \mathfrak{bn}$) etiam

$$\mathfrak{gt}\ |||\ \mathfrak{bn}.$$

Pari modo demonstratur $\mathfrak{gt}\ |||\ \mathfrak{cp}$ esse. Interim $\mathfrak{gt}$ bisecat rectam $\mathfrak{bc}$ perpendiculariter; adeoque $\mathfrak{tgbn} = \mathfrak{tgcp}$ (§. 1.) et

$$\mathfrak{bn}\ |||\ \bumpeq\ \mathfrak{cp}.$$

Si $\mathfrak{bn}$, $\mathfrak{am}$, $\mathfrak{cp}$ in plano sint; sit (*extra* hoc planum cadens) $\mathfrak{fs}\ |||\ \bumpeq\ \mathfrak{am}$; tum (per præc.) $\mathfrak{fs}\ |||\ \bumpeq$ tam ad $\mathfrak{bn}$ quam ad $\mathfrak{cp}$, adeoque et $\mathfrak{bn}\ |||\ \bumpeq\ \mathfrak{cp}$.

§. 11.

Complexus puncti $\mathfrak{a}$, atque *omnium* punctorum, quorum quodvis $\mathfrak{b}$ tale est, ut si $\mathfrak{bn}\ |||\ \mathfrak{am}$ sit, sit etiam $\mathfrak{bn} \bumpeq \mathfrak{am}$; dicatur F: sectio vero ipsius F cum quovis plano rectam $\mathfrak{am}$ complectente nominetur L.

In quavis recta, quæ $|||\ \mathfrak{am}$ est, F gaudet puncto, et nonnisi uno; atque patet L per $\mathfrak{am}$ dividi in duas partes congruentes; dicatur $\mathfrak{a}\tilde{\mathfrak{m}}$ *axis* ipsius L; patet etiam, in quovis plano rectam $\mathfrak{am}$ complectente, pro *axe* $\mathfrak{a}\tilde{\mathfrak{m}}$ unicum L dari. Quodvis eiusmodi L, dicatur L *ipsius* $\mathfrak{a}\tilde{\mathfrak{m}}$ (in plano, de quo agitur, intelligendo). Patet per L circa $\mathfrak{am}$ revolutum, F describi, cuius $\mathfrak{a}\tilde{\mathfrak{m}}$ *axis* vocetur, et vicissim F *axi* $\mathfrak{a}\tilde{\mathfrak{m}}$ *attribuatur.*

§. 12.

Si $\mathfrak{b}$ *ubivis in* L *ipsius* $\mathfrak{a}\tilde{\mathfrak{m}}$ *fuerit, et* $\mathfrak{bn}\ |||\ \bumpeq\ \mathfrak{am}$ (§. 11.); *tum* L *ipsius* $\mathfrak{a}\tilde{\mathfrak{m}}$ *et* L *ipsius* $\mathfrak{b}\tilde{\mathfrak{n}}$ *coincidunt.*

Nam dicatur L ipsius $\mathfrak{b}\tilde{\mathfrak{n}}$ distinctionis ergo l; sitque $\mathfrak{c}$ ubivis in l, et $\mathfrak{cp} \, ||| \bumpeq \mathfrak{bn}$ (§. 11.); erit (cum et $\mathfrak{bn} \, ||| \bumpeq \mathfrak{am}$ sit) $\mathfrak{cp} \, ||| \bumpeq \mathfrak{am}$ (§. 10.), adeoque $\mathfrak{c}$ etiam in L cadet. Et si $\mathfrak{c}$ ubivis in L sit, et $\mathfrak{cp} \, ||| \bumpeq \mathfrak{am}$; tum $\mathfrak{cp} \, ||| \bumpeq \mathfrak{bn}$ (§. 10.); caditque $\mathfrak{c}$ etiam in l (§. 11.). Itaque L et l sunt eadem; ac quævis $\mathfrak{b}\tilde{\mathfrak{n}}$ est etiam axis ipsius L, et inter omnes axes ipsius L, $\bumpeq$ est.

Idem de F eodem modo patet.

§. 13.

Fig. 8. *Si* $\mathfrak{bn} \, ||| \, \mathfrak{am}$, $\mathfrak{cp} \, ||| \, \mathfrak{dq}$, *et* $\mathfrak{bam}+\mathfrak{abn}=2R$ *sit*; *tum etiam* $\mathfrak{dcp}+\mathfrak{cdq}=2R$.

Sit enim $\mathfrak{ea}=\mathfrak{eb}$ et $\mathfrak{efm}=\mathfrak{dcp}$ (§. 4.); erit (cum

$$\mathfrak{bam}+\mathfrak{abn}=2R=\mathfrak{abn}+\mathfrak{abg}$$

sit)

$$\mathfrak{ebg}=\mathfrak{eaf};$$

adeoque si etiam $\mathfrak{bg}=\mathfrak{af}$ sit,

$$\triangle\,\mathfrak{ebg}\equiv\triangle\,\mathfrak{eaf},\quad \mathfrak{beg}=\mathfrak{aef},$$

cadetque $\mathfrak{g}$ in $\mathfrak{f}\tilde{\mathfrak{e}}$. Est porro $\mathfrak{gfm}+\mathfrak{fgn}=2R$ (quia $\mathfrak{egb}=\mathfrak{efa}$). Est etiam $\mathfrak{gn} \, ||| \, \mathfrak{fm}$ (§. 6.); itaque si $\mathfrak{mfrs}\equiv\mathfrak{pcdq}$, tum $\mathfrak{rs} \, ||| \, \mathfrak{gn}$ (§. 7.), et $\mathfrak{r}$ in vel extra $\mathfrak{fg}$ cadit (si $\mathfrak{cd}$ non $=\mathfrak{fg}$, ubi res iam patet).

I. In casu primo est $\mathfrak{frs}$ non $>(2R-\mathfrak{rfm}=\mathfrak{fgn})$, quia $\mathfrak{rs} \, ||| \, \mathfrak{fm}$; ast cum $\mathfrak{rs} \, ||| \, \mathfrak{gn}$ sit, est etiam $\mathfrak{frs}$ non $<\mathfrak{fgn}$; adeoque $\mathfrak{frs}=\mathfrak{fgn}$, et

$$\mathfrak{rfm}+\mathfrak{frs}=\mathfrak{gfm}+\mathfrak{fgn}=2R.$$

Itaque et $\mathfrak{dcp}+\mathfrak{cdq}=2R$.

II. Si $\mathfrak{r}$ extra $\mathfrak{fg}$ cadat; tunc $\mathfrak{ngr}=\mathfrak{mfr}$, sitque $\mathfrak{mfgn}\equiv\mathfrak{nghl}\equiv\mathfrak{lhko}$ et ita porro, usquequo $\mathfrak{fk}=$ vel prima vice $>\mathfrak{fr}$ fiat. Est heic $\mathfrak{ko} \, ||| \, \mathfrak{hl} \, ||| \, \mathfrak{fm}$ (§. 7.). Si $\mathfrak{k}$ in $\mathfrak{r}$ cadat; tum $\mathfrak{ko}$ in $\mathfrak{rs}$ cadit (§. 1.); adeoque

$$\mathfrak{rfm}+\mathfrak{frs}=\mathfrak{kfm}+\mathfrak{fko}=\mathfrak{kfm}+\mathfrak{fgn}=2R;$$

si vero $\mathfrak{r}$ in $\mathfrak{hk}$ cadat, tum (per I.) est

$$\mathfrak{rhl}+\mathfrak{hrs}=2R=\mathfrak{rfm}+\mathfrak{frs}=\mathfrak{dcp}+\mathfrak{cdq}.$$

§. 14.

Si bn ||| am, cp ||| dq, *et* bam+abn$<2R$ *sit*; *tum etiam* dcp+cdq$<2R$.

Si enim dcp+cdq non esset $<$, adeoque (per §. 1.) esset $=2R$; tum (per §. 13.) etiam bam+abn$=2R$ esset (contra hyp.).

§. 15.

Perpensis §§. 13. et 14. *Systema Geometriae hypothesi veritatis Axiomatis Euclidei XI. insistens dicatur* Σ; *et hypothesi contrariae superstructum sit S. Omnia, quae expresse non dicentur, in* Σ *vel in S esse; absolute enuntiari*, i. e. *illa, sive* Σ *sive S reipsa sit, vera asseri intelligatur.*

§. 16.

Si am *sit axis alicuius L; tum L in* Σ *recta* $\perp$ am *est.* Fig. 5.

Nam sit e quovis puncto b ipsius L axis bn; erit in Σ

$$\text{bam}+\text{abn}=2\text{bam}=2R,$$

adeoque bam$=R$. Et si c quodvis punctum in $\widetilde{\text{ab}}$ sit, atque cp ||| am; est (per §. 13.) cp ≏ am, adeoque c in L (§. 11.).

In S vero nulla 3 *puncta* a, b, c *ipsius L vel F in recta sunt.*

Nam aliquis axium am, bn, cp (ex. gr. am) intra duos reliquos cadit; et tunc (per §. 14.) tam bam quam cam$<R$.

§. 17.

L est etiam in S linea, et F superficies. Fig. 7.

Nam (per §. 11.) quodvis planum ad axem am̃ (per punctum aliquod ipsius F) perpendiculare secat ipsum F in peripheria circuli, cuius planum (per §. 14.) ad nullum alium axem bñ perpendiculare est. Revolvatur F circa bn; manebit (per §. 12.) quodvis punctum ipsius F in F, et sectio ipsius F cum plano ad bñ non perpendiculari describet super-

ficiem: atqui F (per §. 12.), quæcunque puncta $\mathfrak{a}$, $\mathfrak{b}$ fuerint in eo, ita *sibi* congruere poterit, ut $\mathfrak{a}$ in $\mathfrak{b}$ cadat; est igitur F *superficies uniformis.*

Patet hinc (per §§. 11. et 12.) L esse *lineam uniformem.**

§. 18.

Fig. 7. *Cuiusvis plani, per punctum* $\mathfrak{a}$ *ipsius* F *ad axem* $\mathfrak{am}$ *oblique positi, sectio cum* F *in* S *peripheria circuli est.*

Nam sint $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ 3 puncta huius sectionis, et $\mathfrak{bn}$, $\mathfrak{cp}$ axes; facient $\mathfrak{ambn}$, $\mathfrak{amcp}$ angulum; nam secus planum (ex §. 16.) per $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ determinatum ipsam $\mathfrak{am}$ complecteretur (contra hyp.). Plana igitur, rectas $\mathfrak{ab}$, $\mathfrak{ac}$ perpendiculariter bisecantia se mutuo secant (§. 10.) in aliquo axe $\mathfrak{f}\bar{\mathfrak{s}}$ (ipsius F), atque $\mathfrak{fb} = \mathfrak{fa} = \mathfrak{fc}$. Sit $\mathfrak{ah} \perp \mathfrak{fs}$, et revolvatur $\mathfrak{fah}$ circa $\mathfrak{fs}$; describet $\mathfrak{a}$ peripheriam radii $\mathfrak{ha}$, per $\mathfrak{b}$ et $\mathfrak{c}$ euntem, et *simul* in F et $\overset{\frown}{\mathfrak{abc}}$ sitam; nec F et $\overset{\frown}{\mathfrak{abc}}$ præter $\bigcirc\,\mathfrak{ha}$ quidquam commune habent (§. 16.).

Patet etiam extremitate portionis $\mathfrak{fa}$ lineæ L (tanquam radio) in F circa $\mathfrak{f}$ mota ipsam $\bigcirc\,\mathfrak{ha}$ describi.

§. 19.

Fig. 5. *Perpendicularis* $\mathfrak{bt}$ *ad axem* $\mathfrak{bn}$ *ipsius* L (in planum ipsius L cadens) *est in* S *tangens ipsius* L.

Nam L in $\mathfrak{b}\tilde{\mathfrak{t}}$ præter $\mathfrak{b}$ nullo puncto gaudet (§. 14.), si vero $\mathfrak{bq}$ in $\mathfrak{tbn}$ cadat, tum centrum sectionis plani per $\mathfrak{bq}$ ad $\mathfrak{tbn}$ perpendicularis cum F ipsius $\mathfrak{b}\tilde{\mathfrak{n}}$ (§. 18.) manifesto in $\mathfrak{b}\tilde{\mathfrak{q}}$ locatur, et si $\mathfrak{bq}$ diameter sit, patet $\mathfrak{b}\tilde{\mathfrak{q}}$ lineam L ipsius $\mathfrak{b}\tilde{\mathfrak{n}}$ in $\mathfrak{q}$ secare.

§. 20.

Per quaevis duo puncta in F *linea* L *determinatur* (§§. 11. et 18.); atque (cum ex §§. 16. et 19. L perpendicularis ad omnes suos axes sit)

* Demonstrationem ad S restringere haud necesse est; quum facile ita proponatur, ut absolute (pro S et Σ) valeat. (Ed. I. Tom. I. Errata Appendicis).

quivis angulus Llineus in F angulo planorum ad F per crura perpendicularium aequalis est.

§. 21.

Duae lineae Lformes ap̄, bδ̄ *in eodem F, cum tertia Lformi* ab *summam internorum* $< 2R$ *efficientes, se mutuo secant* (per ap̄ in F intelligendo L per a, p ductum, per ap̄ vero dimidium illud eius ex a incipiens, in quod p cadit). Fig. 6.

Nam si am, bn axes ipsius F sint; tum amp̄, bnδ̄ secant se invicem (§. 9.); atque F secat eorundem sectionem (per §§. 7. et 11.); adeoque et ap̄, bδ̄ se mutuo secant.

Patet exhinc Axioma XI. et omnia, quæ in Geometria Trigonometriaque (plana) asseruntur, *absolute* constare in F, rectarum vices lineis L subeuntibus: idcirco functiones trigonometricæ abhinc eodem sensu accipientur, quo in Σ veniunt; et peripheria circuli, cuius radius Lformis $= r$ in F, est $= 2\pi r$, et pariter $\odot r$ (in F) $= \pi r^2$ (per π intelligendo $\frac{1}{2} \bigcirc 1$ in F, sive notum $3{\cdot}1415926\ldots$).

§. 22.

Si ab̄ *fuerit L ipsius* am̄, *et* c *in* am̄; *atque angulus* cab (e recta am̄ et Lformi linea ab̄ compositus) *feratur prius iuxta* ab̄, *tum iuxta* bā *semper porro in infinitum: erit via* cδ̄ *ipsius* c *linea L ipsius* cm̄. Fig. 9.

Nam (posteriore l dicta) sit punctum quodvis δ in cδ̄, δn ||| cm, et b punctum ipsius L in δ̄n cadens; erit bn ≏ am, et ac = bδ, adeoque δn ≏ cm, consequ. δ in l. Si vero δ in l et δn ||| cm, atque b punctum ipsius L ipsi δ̄n commune sit; erit am ≏ bn et cm ≏ δn, unde manifesto bδ = ac, cadetque δ in viam puncti c, et sunt l et cδ̄ eadem. Designetur tale l per $l \parallel L$.

§. 23.

Si linea Lformis cδf || abe (§. 22.), et ab = be, atque am̄, bn̄, ep̄ sint axes; erit manifesto cδ = δf; et si quælibet 3 puncta a, b, e fuerint ipsius Fig. 9.

$\widetilde{\mathfrak{ab}}$, ac $\mathfrak{ab} = n . \mathfrak{cd}$: erit quoque $\mathfrak{ae} = n . \mathfrak{cf}$; adeoque (manifesto etiam pro $\mathfrak{ab}$, $\mathfrak{ae}$, $\mathfrak{dc}$ incommensurabilibus)

$$\mathfrak{ab} : \mathfrak{cd} = \mathfrak{ae} : \mathfrak{cf},$$

estque $\mathfrak{ab} : \mathfrak{cd}$ ab $\mathfrak{ab}$ *independens* et per $\mathfrak{ac}$ *prorsus determinatum*. Denotetur quotus iste, nempe $\mathfrak{ab} : \mathfrak{cd}$ litera maiore eiusdem nominis (puta per X), quo $\mathfrak{ac}$ litera minuscula (ex. gr. x) insignitur.

§. 24.

Quaecunque x et y fuerint; *est* $Y = X^{\frac{y}{x}}$ (§. 23.).

Nam aut erit alterum (ipsorum x, y) multiplum alterius (ex. gr. y ipsius x), aut non.

Si $y = nx$; sit $x = \mathfrak{ac} = \mathfrak{cg} = \mathfrak{gh}$ &, usquequo $\mathfrak{ah} = y$ fiat; sit porro $\mathfrak{cd} \parallel \mathfrak{gk} \parallel \mathfrak{hl}$; erit (§. 23.)

$$X = \mathfrak{ab} : \mathfrak{cd} = \mathfrak{cd} : \mathfrak{gk} = \mathfrak{gk} : \mathfrak{hl};$$

adeoque

$$\frac{\mathfrak{ab}}{\mathfrak{hl}} = \left(\frac{\mathfrak{ab}}{\mathfrak{cd}}\right)^n,$$

sive

$$Y = X^n = X^{\frac{y}{x}}.$$

Si x, y multipla ipsius i sint, puta

$$x = mi \quad \text{et} \quad y = ni;$$

est (per præc.)

$$X = I^m, \; Y = I^n,$$

consequ.

$$Y = X^{\frac{n}{m}} = X^{\frac{y}{x}}.$$

Idem ad casum incommensurabilitatis ipsorum x, y facile extenditur. Si vero fuerit $q = y - x$; erit manifesto $Q = Y : X$.

Nec non manifestum est, in Σ pro quovis x esse $X = 1$, in S vero $X > 1$ esse, atque pro *quibusvis* $\mathfrak{ab}$, $\mathfrak{abe}$ dari tale $\mathfrak{cdf} \parallel \mathfrak{abe}$, ut sit $\mathfrak{cdf} = \mathfrak{ab}$,

unde $\mathfrak{ambn} = \mathfrak{amep}$ erit, etsi hoc illius qualevis multiplum sit; quod singulare quidem est, sed absurditatem ipsius S evidenter non probat.

§. 25.

In quovis rectilineo triangulo sunt peripheriae radiorum lateribus aequalium, uti sinus angulorum oppositorum. Fig. 10.

Sit enim $\mathfrak{abc} = R$, et $\mathfrak{am} \perp \mathfrak{bac}$, atque sint $\mathfrak{bn}$, $\mathfrak{cp} \parallel \mathfrak{am}$; erit $\mathfrak{cab} \perp \mathfrak{ambn}$, adeoque (cum $\mathfrak{cb} \perp \mathfrak{ba}$ sit) $\mathfrak{cb} \perp \mathfrak{ambn}$, consequ. $\mathfrak{cpbn} \perp \mathfrak{ambn}$. Secet F *ipsius* $\overline{\mathfrak{cp}}$ rectas $\overline{\mathfrak{bn}}$, $\overline{\mathfrak{am}}$ (respective) in $\mathfrak{d}$, $\mathfrak{e}$, et fascias $\mathfrak{cpbn}$, $\mathfrak{cpam}$, $\mathfrak{bnam}$ in lineis L formibus $\mathfrak{cd}$, $\mathfrak{ce}$, $\mathfrak{de}$; erit (§. 20.) $\mathfrak{cde}$ = angulo ipsorum $\mathfrak{ndc}$, $\mathfrak{nde}$, adeoque $= R$; atque pari ratione est $\mathfrak{ced} = \mathfrak{cab}$.

Est autem (per §. 21.) in L lineo triangulo $\mathfrak{ced}$ (heic radio semper $=1$ posito)

$$\mathfrak{ec} : \mathfrak{dc} = 1 : \sin.\mathfrak{dec} = 1 : \sin.\mathfrak{cab}.$$

Est quoque (per §. 21.)

$$\mathfrak{ec} : \mathfrak{dc} = \bigcirc\,\mathfrak{ec} : \bigcirc\,\mathfrak{dc} \ (\text{in } F)$$
$$= \bigcirc\,\mathfrak{ac} : \bigcirc\,\mathfrak{bc} \ (§.\ 18.);$$

adeoque est etiam

$$\bigcirc\,\mathfrak{ac} : \bigcirc\,\mathfrak{bc} = 1 : \sin.\mathfrak{cab};$$

unde assertum pro quovis triangulo liquet.

§. 26.

In quovis sphaerico triangulo sunt sinus laterum, uti sinus angulorum iisdem oppositorum. Fig. 11.

Nam sit $\mathfrak{abc} = R$, et $\mathfrak{ced}$ perpendiculare ad sphæræ radium $\mathfrak{oa}$; erit $\mathfrak{ced} \perp \mathfrak{aob}$, et (cum etiam $\mathfrak{boc} \perp \mathfrak{boa}$ sit) $\mathfrak{cd} \perp \mathfrak{ob}$. In triangulis $\mathfrak{ceo}$, $\mathfrak{cdo}$ vero est (per §. 25.)

$$\bigcirc\,\mathfrak{ec} : \bigcirc\,\mathfrak{oc} : \bigcirc\mathfrak{dc} = \sin.\mathfrak{coe} : 1 : \sin.\mathfrak{cod}$$
$$= \sin.\mathfrak{ac} : 1 : \sin.\mathfrak{bc};$$

interim (§. 25.) etiam

$$\bigcirc \mathfrak{ec} : \bigcirc \mathfrak{dc} = \sin.\ \mathfrak{cde} : \sin.\ \mathfrak{ced};$$

itaque

$$\sin.\ \mathfrak{ac} : \sin.\ \mathfrak{bc} = \sin.\ \mathfrak{cde} : \sin.\ \mathfrak{ced};$$

est vero $\mathfrak{cde} = R = \mathfrak{cba}$, atque $\mathfrak{ced} = \mathfrak{cab}$. Consequenter

$$\sin.\ \mathfrak{ac} : \sin.\ \mathfrak{bc} = 1 : \sin.\ \mathfrak{a}.$$

E quo promanans Trigonometria sphaerica ab Axiomate XI. independenter stabilita est.

§. 27.

Fig. 12. *Si* $\mathfrak{ac}$, $\mathfrak{bd}$ *sint* $\perp \mathfrak{ab}$, *et feratur* $\mathfrak{cab}$ *iuxta* $\widetilde{\mathfrak{ab}}$; *erit* (*via puncti* $\mathfrak{c}$ *dicta heic* $\mathfrak{cd}$)

$$\mathfrak{cd} : \mathfrak{ab} = \sin.\ u : \sin.\ v.$$

Nam sit $\mathfrak{de} \perp \mathfrak{ca}$; est in triangulis $\mathfrak{ade}$, $\mathfrak{adb}$ (per §. 25.)

$$\bigcirc \mathfrak{ed} : \bigcirc \mathfrak{ad} : \bigcirc \mathfrak{ab} = \sin.\ u : 1 : \sin.\ v.$$

Revoluto $\mathfrak{bacd}$ circa $\mathfrak{ac}$, describetur $\bigcirc \mathfrak{ab}$ per $\mathfrak{b}$, $\bigcirc \mathfrak{ed}$ per $\mathfrak{d}$; et via dictæ $\mathfrak{cd}$ denotetur heic per $\odot \mathfrak{cd}$. Sit porro polygonum quodvis $\mathfrak{bfg} \ldots$ ipsi $\odot \mathfrak{ab}$ inscriptum; nascetur per plana ex omnibus lateribus $\mathfrak{bf}$, $\mathfrak{fg}$ &, ad $\odot \mathfrak{ab}$ perpendicularia, in $\odot \mathfrak{cd}$ quoque figura polygonalis totidem laterum; et demonstrari (ad instar §. 23.) potest, esse

$$\mathfrak{cd} : \mathfrak{ab} = \mathfrak{dh} : \mathfrak{bf} = \mathfrak{hk} : \mathfrak{fg} = \cdots,$$

adeoque

$$\mathfrak{dh} + \mathfrak{hk} + \cdots : \mathfrak{bf} + \mathfrak{fg} + \cdots = \mathfrak{cd} : \mathfrak{ab}$$

Quovis laterum $\mathfrak{bf}$, $\mathfrak{fg}$, ... ad limitem o tendente, manifesto

$$\mathfrak{bf} + \mathfrak{fg} + \cdots \rightharpoondown \bigcirc \mathfrak{ab} \quad \text{et} \quad \mathfrak{dh} + \mathfrak{hk} + \cdots \rightharpoondown \bigcirc \mathfrak{ed}.$$

Itaque etiam

$$\bigcirc \mathfrak{ed} : \bigcirc \mathfrak{ab} = \mathfrak{cd} : \mathfrak{ab}.$$

Erat vero

$$\bigcirc \mathfrak{ed} : \bigcirc \mathfrak{ab} = \sin.\ u : \sin.\ v.$$

Consequ.

$$\mathfrak{cd} : \mathfrak{ab} = \sin.\ u : \sin.\ v.$$

Remoto ac a bd in infinitum, manet

$$\text{cd}:\text{ab}$$

adeoque etiam

$$\sin. u : \sin. v$$

constans; u vero $\rightharpoondown R$ (§. 1.), et si dm ||| bn sit, $v \rightharpoondown z$; unde fit

$$\text{cd} : \text{ab} = 1 : \sin. z.$$

Via dicta cd denotabitur per cd || ab.

§. 28.

Si bn ||| ≏ am, *et* c *in* am̃, *atque* ac $=x$ *sit: erit* X (§. 23.) Fig. 13.

$$= \sin. u : \sin. v.$$

Nam si cd et ae sint ⊥ bn et bf ⊥ am; erit (ad instar §. 27.)

$$\bigcirc\, \text{bf} : \bigcirc\, \text{cd} = \sin. u : \sin. v.$$

Est autem evidenter bf $=$ ae: quamobrem

$$\bigcirc\, \text{ea} : \bigcirc\, \text{dc} = \sin. u : \sin. v.$$

In superficiebus vero Fformibus ipsorum am et cm (ipsum ambn in ab et cg secantibus) est (per §. 21.)

$$\bigcirc\, \text{ea} : \bigcirc\, \text{dc} = \text{ab} : \text{cg} = X.$$

Est itaque etiam

$$X = \sin. u : \sin. v.$$

§. 29.

Si bam $= R$, ab $= y$, *et* bn ||| am *sit*; *erit in* S Fig. 14.

$$Y = \cot. \frac{1}{2} u.$$

Nam si fuerit ab=ac, et cp ||| am (adeoque bn ||| ≏ cp), atque pcd=qcd; datur (§. 19.) ds ⊥ $\overline{\text{cd}}$, ut ds ||| cp, adeoque (§. 1.) dt ||| cq sit. Si porro be ⊥ $\overline{\text{ds}}$; erit (§. 7.) ds ||| bn, adeoque (§. 6.) bn ||| es, et (cum dt ||| cq sit) bq ||| et; consequ. (§. 1.) ebn = ebq.

Repræsententur, bcf ex L ipsius bn, et fg, dh, ck et el ex L formibus lineis ipsorum ft, dt, cq et et; erit evidenter (§. 22.)

$$\text{hg} = \text{df} = \text{dk} = \text{hc};$$

itaque

$$\text{cg} = 2\text{ch} = 2v.$$

Pariter patet

$$\text{bg} = 2\text{bl} = 2z$$

esse. Est vero

$$\text{bc} = \text{bg} - \text{cg};$$

quapropter

$$y = z - v,$$

adeoque (§. 24.)

$$Y = Z : V.$$

Est demum (§. 28.)

$$Z = 1 : \sin.\frac{1}{2}u \quad \text{et} \quad V = 1 : \sin.\left(R - \frac{1}{2}u\right),$$

consequ.

$$Y = \cot.\frac{1}{2}u.$$

§. 30.

Fig. 15. Verumtamen facile (ex §. 25.) patet, resolutionem problematis *Trigonometriae planae* in S, peripheriæ per radium expressæ indigere; hoc vero rectificatione ipsius L obtineri potest.

Sint ab, cm, c'm' ⊥ $\overline{\text{ac}}$, atque b ubivis in $\overline{\text{ab}}$; erit (§. 25.)

$$\sin.u : \sin.v = \bigcirc p : \bigcirc y$$

et

$$\sin.u' : \sin.v' = \bigcirc p : \bigcirc y';$$

adeoque

$$\frac{\sin.u}{\sin.v}\bigcirc y = \frac{\sin.u'}{\sin.v'}\bigcirc y'.$$

Est vero (per §. 27.)

$$\sin. v : \sin. v' = \cos. u : \cos. u';$$

consequ.

$$\frac{\sin. u}{\cos. u} \bigcirc y = \frac{\sin. u'}{\cos. u'} \bigcirc y',$$

seu

$$\bigcirc y : \bigcirc y' = \text{tang.}\, u' : \text{tang.}\, u = \text{tang.}\, w : \text{tang.}\, w'.$$

Sint porro cn ||| ab, c'n' ||| ab et cd, c'd' lineæ L formes ad $\overline{ab}$ perpendiculares; erit (§. 21.) etiam

$$\bigcirc y : \bigcirc y' = r : r',$$

adeoque

$$r : r' = \text{tang.}\, w : \text{tang.}\, w'.$$

Crescat iam p ab a incipiendo in infinitum; tum $w \sim z$ et $w' \sim z'$; quapropter etiam

$$r : r' = \text{tang.}\, z : \text{tang.}\, z'.$$

Constans $r : \text{tang.}\, z$ (ab r *independens*) dicatur i; dum $y \sim 0$, est

$$\left(\frac{r}{y} = \frac{i\,\text{tang.}\, z}{y}\right) \sim 1,$$

adeoque

$$\frac{y}{\text{tang.}\, z} \sim i.$$

Ex §. 29. fit

$$\text{tang.}\, z = \frac{1}{2}(Y - Y^{-1});$$

itaque

$$\frac{2y}{Y - Y^{-1}} \sim i,$$

seu (§. 24.)

$$\frac{2yI^{\frac{y}{i}}}{I^{\frac{2y}{i}} - 1} \sim i.$$

Notum autem est, expressionis istius (dum $y \sim 0$) limitem esse $\frac{i}{\log. \text{nat.}\, I}$; est ergo

$$\frac{i}{\log. \text{nat.}\, I} = i \quad \text{et} \quad I = e = 2{\cdot}7182818\ldots,$$

quæ quantitas insignis hic quoque elucet. Si nempe abhinc i illam rectam denotet, cuius $I = e$ sit, erit $r = i$ tang. z. Erat autem (§. 21.) $\bigcirc y = 2\pi r$; est igitur

$$\bigcirc y = 2\pi i \text{ tang. } z = \pi i\,(Y - Y^{-1}) =$$
$$= \pi i\,(e^{\frac{y}{i}} - e^{-\frac{y}{i}}) = \frac{\pi y}{\text{log. nat. } Y}\,(Y - Y^{-1})$$

(per §. 24.).

§. 31.

Fig. 16. Ad resolutionem omnium triangulorum rectangulorum rectilineorum trigonometricam (e qua omnium triangulorum resolutio in promtu est) in S 3 æquationes sufficiunt: nempe (a, b cathetos, c hypotenusam, et α, β angulos cathetis oppositos denotantibus) æquatio relationem exprimens *primo* inter a, c, α, *secundo* inter a, α, β, *tertio* inter a, b, c; nimirum ex his *reliquae* 3 per eliminationem prodeunt.

I. Ex §§. 25. et 30. est

$$1 : \sin.\alpha = (C - C^{-1}) : (A - A^{-1}) = (e^{\frac{c}{i}} - e^{-\frac{c}{i}}) : (e^{\frac{a}{i}} - e^{-\frac{a}{i}});$$

(æquatio pro α, c, a).

II. Ex §. 27. sequitur (si $\beta\mathfrak{m} \,\|\, \gamma\mathfrak{n}$ sit)

$$\cos.\alpha : \sin.\beta = 1 : \sin. u;$$

ex §. 29. autem fit

$$1 : \sin. u = \frac{1}{2}(A + A^{-1});$$

itaque

$$\cos.\alpha : \sin.\beta = \frac{1}{2}(A + A^{-1}) = \frac{1}{2}(e^{\frac{a}{i}} + e^{-\frac{a}{i}});$$

(æquatio pro α, β, a).

III. Si $\alpha\alpha' \perp \beta\alpha\gamma$, atque $\beta\beta'$ et $\gamma\gamma'$ fuerint $\|\, \alpha\alpha'$, (§. 27.), atque $\beta'\alpha'\gamma' \perp \alpha\alpha'$; erit manifesto (uti in §. 27.)

$$\frac{\beta\beta'}{\gamma\gamma'} = \frac{1}{\sin. u} = \frac{1}{2}(A + A^{-1}),$$
$$\frac{\gamma\gamma'}{\alpha\alpha'} = \frac{1}{2}(B + B^{-1}),$$

ac

$$\frac{\beta\beta'}{\alpha\alpha'} = \frac{1}{2}(C + C^{-1});$$

consequ.

$$\frac{1}{2}(C + C^{-1}) = \frac{1}{2}(A + A^{-1})\frac{1}{2}(B + B^{-1}),$$

sive

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \frac{1}{2}\left(e^{\frac{a}{i}} + e^{-\frac{a}{i}}\right)\left(e^{\frac{b}{i}} + e^{-\frac{b}{i}}\right);$$

(æquatio pro a, b, c).

Si $\gamma\alpha\delta = R$, et $\beta\delta \perp \alpha\delta$ sit; erit

$$\bigcirc c : \bigcirc a = 1 : \sin.\alpha,$$

et

$$\bigcirc c : \bigcirc (d = \beta\delta) = 1 : \cos.\alpha,$$

adeoque ($\bigcirc x^2$ pro quovis x factum $\bigcirc x . \bigcirc x$ denotante) manifesto

$$\bigcirc a^2 + \bigcirc d^2 = \bigcirc c^2.$$

Est vero (per §. 27. et II.)

$$\bigcirc d = \bigcirc b . \frac{1}{2}(A + A^{-1}),$$

consequ.

$$\left(e^{\frac{c}{i}} - e^{-\frac{c}{i}}\right)^2 = \frac{1}{4}\left(e^{\frac{a}{i}} + e^{-\frac{a}{i}}\right)^2\left(e^{\frac{b}{i}} - e^{-\frac{b}{i}}\right)^2 + \left(e^{\frac{a}{i}} - e^{-\frac{a}{i}}\right)^2;$$

alia æquatio pro a, b, c (cuius membrum *secundum* facile ad formam *symmetricam* seu *invariabilem* reducitur).

Denique ex

$$\frac{\cos.\alpha}{\sin.\beta} = \frac{1}{2}(A + A^{-1})$$

atque

$$\frac{\cos.\beta}{\sin.\alpha} = \frac{1}{2}(B + B^{-1})$$

fit (per III.)

$$\cot.\alpha \cot.\beta = \frac{1}{2}\left(e^{\frac{c}{i}} + e^{-\frac{c}{i}}\right);$$

(æquatio pro α, β, c).

§. 32.

Restat adhuc modum *problemata* in S resolvendi breviter ostendere, quo (per exempla magis obvia) peracto, demum quid theoria hæcce præstet, candide dicetur.

Fig. 17. I. Sit $\overset{\frown}{\mathfrak{ab}}$ linea in plano, et $y=f(x)$ æquatio eius (pro coordinatis perpendicularibus), et quodvis incrementum ipsius z dicatur dz, atque incrementa ipsorum x, y, et areæ u, eidem dz respondentia, respective per dx, dy, du denotentur; sitque $\mathfrak{bh} \parallel \mathfrak{cf}$, et exprimatur (ex §§. 31. et 27.) $\frac{\mathfrak{bh}}{dx}$ per y, ac quæratur ipsius $\frac{dy}{dx}$ *limes* tendente dx ad limitem 0, (quod, ubi eiusmodi limes quæritur, subintelligatur): innotescet exinde etiam limes ipsius $\frac{dy}{\mathfrak{bh}}$, adeoque tg. $\mathfrak{hbg}$; eritque, (cum $\mathfrak{hbc}$ manifesto nec $>$ nec $<$ adeoque $= R$ sit), *tangens* in $\mathfrak{b}$ ipsius $\mathfrak{bg}$ per y determinata.

II. Demonstrari potest, esse

$$\frac{dz^2}{dy^2+\mathfrak{bh}^2} \sim 1.$$

Hinc *limes* ipsius $\frac{dz}{dx}$, et inde z integratione (per x expressum) reperitur.

Et potest lineæ cuiusvis *in concreto datae* æquatio in S inveniri, ex. gr. ipsius L.

Si enim $\mathfrak{am}$ axis ipsius L sit; tum quævis $\overset{\frown}{\mathfrak{cb}}$ ex $\mathfrak{am}$ secat L (cum per §. 19 quævis recta ex $\mathfrak{a}$ præter $\mathfrak{am}$ ipsum L secet); est vero (si $\mathfrak{bn}$ axis sit)

$$X = 1 : \sin. \mathfrak{cbn} \quad (§. 28.),$$

atque

$$Y = \cot. \frac{1}{2}\, \mathfrak{cbn} \quad (§. 29.),$$

unde fit

$$Y = X + \sqrt{X^2 - 1}$$

seu

$$e^{\frac{y}{i}} = e^{\frac{x}{i}} + \sqrt{e^{\frac{2x}{i}} - 1}$$

æquatio quæsita. Erit hinc

$$\frac{dy}{dx} \backsim X(X^2-1)^{-\frac{1}{2}};$$

atqui

$$\frac{\mathfrak{bh}}{dx} = 1 : \sin.\, \mathfrak{cbn} = X;$$

adeoque

$$\frac{dy}{\mathfrak{bh}} \backsim (X^2-1)^{-\frac{1}{2}};$$

$$1 + \frac{dy^2}{\mathfrak{bh}^2} \backsim X^2(X^2-1)^{-1},$$

$$\frac{dz^2}{\mathfrak{bh}^2} \backsim X^2(X^2-1)^{-1},$$

$$\frac{dz}{\mathfrak{bh}} \backsim X(X^2-1)^{-\frac{1}{2}}$$

atque

$$\frac{dz}{dx} \backsim X^2(X^2-1)^{-\frac{1}{2}};$$

unde per integrationem invenitur

$$z = i(X^2-1)^{\frac{1}{2}} = i \cot.\, \mathfrak{cbn}$$

(uti §. 30.).

III. Manifesto

$$\frac{du}{dx} \backsim \frac{\mathfrak{hfcbh}}{dx},$$

quod (nonnisi ab y dependens) iam primum per y exprimendum est; unde u integrando prodit.

Si $\mathfrak{ab} = p$, $\mathfrak{ac} = q$, et $\mathfrak{cd} = r$, atque $\mathfrak{cabdc} = s$ sit; poterit (uti in II.) Fig. 12.
ostendi esse

$$\frac{ds}{dq} \backsim r,$$

quod

$$= \frac{1}{2}\, p\left(e^{\frac{q}{i}} + e^{-\frac{q}{i}}\right),$$

atque integrando

$$s = \frac{1}{2}\, pi\left(e^{\frac{q}{i}} - e^{-\frac{q}{i}}\right).$$

Potest hoc absque integratione quoque deduci.

Aequatione e. g. circuli (ex §. 31, III.), rectæ (ex §. 31, II.), sectionis coni (per præc.) expressis; poterunt areæ quoque his lineis clausæ exprimi.

Palam est, superficiem t ad figuram planam p (in distantia q) || lam esse ad p in ratione potentiarum secundarum linearum homologarum, sive uti

$$\frac{1}{4}(e^{\frac{q}{i}}+e^{-\frac{q}{i}})^2:1.$$

Porro computum soliditatis pari modo tractatum, facile patet duas integrationes requirere (cum et differentiale ipsum hic nonnisi per integrationem determinetur); et ante omnia solidum a p et t ac complexu omnium rectarum ad p perpendicularium, fines ipsorum p, t connectentium, clausum quærendum esse. Reperitur solidum istud (tam per integrationem quam sine ea)

$$=\frac{1}{8}pi(e^{\frac{2q}{i}}-e^{-\frac{2q}{i}})+\frac{1}{2}pq.$$

Superficies quoque corporum is S determinari possunt, nec non *curvaturae*, *evolutae*, *evolventesque* linearum qualiumvis &. Quod curvaturam attinet; ea in S aut ipsius L est, aut per radium circuli, aut *distantiam* curvæ ad rectam || læ ab hac recta, determinatur; cum e præcedentibus facile ostendi possit, præter L, lineas circulares, ac rectæ || las, nullas in plano alias lineas uniformes dari.

IV. Pro circulo est (uti in III.)

$$\frac{d\odot x}{dx}\backsim\bigcirc x,$$

unde (per §. 30.) integrando fit

$$\odot x=\pi i^2(e^{\frac{x}{i}}-2+e^{-\frac{x}{i}}).$$

Fig. 9. V. Pro area $\mathfrak{cabdc}=u$ (linea L formi $\mathfrak{ab}=r$, huic || la $\mathfrak{cd}=y$, ac rectis $\mathfrak{ac}$, $\mathfrak{bd}=x$ clausa) est

$$\frac{du}{dx} \backsim y,$$

atque (§. 24.)

$$y = re^{-\frac{x}{i}};$$

adeoque (integrando)

$$u = ri\left(1 - e^{-\frac{x}{i}}\right).$$

Crescente x in infinitum, fiet in S $e^{-\frac{x}{i}} \backsim 0$, adeoque $u \backsim ri$. Per *quantitatem* ipsius $\mathfrak{mabn}$ in posterum limes iste intelligetur.

Simili modo invenitur, quod si p sit figura in F; spatium a p et complexu axium e terminis ipsius p ductorum clausum $= \frac{1}{2} pi$ sit.

VI. Si angulus ad centrum segmenti z sphæræ sit $2u$, peripheria circuli maximi sit p, et arcus $\mathfrak{fc}$ (anguli u) $= x$; erit (§. 25.) Fig. 10.

$$1 : \sin. u = p : \bigcirc \mathfrak{bc},$$

et hinc

$$\bigcirc \mathfrak{bc} = p \sin. u.$$

Interim est

$$x = \frac{pu}{2\pi}, \quad \text{ac} \quad dx = \frac{pdu}{2\pi}.$$

Est porro

$$\frac{dz}{dx} \backsim \bigcirc \mathfrak{bc},$$

et hinc

$$\frac{dz}{du} \backsim \frac{p^2}{2\pi} \sin. u,$$

unde (integrando)

$$z = \frac{\text{sin. vers. } u}{2\pi} p^2.$$

Cogitetur F in quod p (per meditullium $\mathfrak{f}$ segmenti transiens) cadit; planis $\overline{\mathfrak{fem}}$, $\overline{\mathfrak{cem}}$ per $\mathfrak{af}$, $\mathfrak{ac}$ ad F perpendiculariter positis, ipsumque in $\mathfrak{feg}$, $\mathfrak{ce}$ secantibus; et considerentur L formis $\mathfrak{cd}$ (ex $\mathfrak{c}$ ad $\mathfrak{feg}$ perpendicularis) nec non L formis $\mathfrak{cf}$; erit (§. 20.)

$$\mathfrak{cef} = u,$$

et (§. 21.)

$$\frac{\mathfrak{fd}}{p} = \frac{\text{sin. vers. } u}{2\pi},$$

adeoque

$$z = \mathfrak{fd} . p.$$

Ast (§. 21.)

$$p = \pi . \mathfrak{fdg},$$

itaque

$$z = \pi . \mathfrak{fd} . \mathfrak{fdg}.$$

Est autem (§. 21.)

$$\mathfrak{fd} . \mathfrak{fdg} = \mathfrak{fc} . \mathfrak{fc};$$

consequ.

$$z = \pi . \mathfrak{fc} . \mathfrak{fc} = \odot\, \mathfrak{fc} \text{ in } F.$$

Fig. 14. Sit iam $\mathfrak{bj} = \mathfrak{cj} = r$; erit (§. 30.)

$$2r = i\,(Y - Y^{-1}),$$

adeoque (§. 21.)

$$\odot\, 2r \text{ (in } F) = \pi i^2 (Y - Y^{-1})^2.$$

Est quoque (IV.)

$$\odot\, 2y = \pi i^2 (Y^2 - 2 + Y^{-2});$$

igitur $\odot\, 2r$ (in F) $= \odot\, 2y$, adeoque *et superficies z segmenti sphaerici aequatur circulo, chorda* $\mathfrak{fc}$ *tanquam radio descripto.*

Hinc tota sphæræ superficies

$$= \odot\, \mathfrak{fg} = \mathfrak{fdg} . p = \frac{p^2}{\pi},$$

suntque superficies sphaerarum, uti secundae potentiae peripheriarum earundem maximarum.

VII. Soliditas sphæræ radii x in S reperitur simili modo

$$= \frac{1}{2}\, \pi i^3 (X^2 - X^{-2}) - 2\pi i^2 x;$$

Fig. 12. superficies per revolutionem lineæ $\mathfrak{cd}$ circa $\mathfrak{ab}$ orta

$$= \frac{1}{2}\, \pi i p\, (Q^2 - Q^{-2}),$$

et corpus per $\mathfrak{cabdc}$ descriptum

$$= \frac{1}{4}\, \pi i^2 p\, (Q - Q^{-1})^2.$$

Quomodo vero omnia a (IV.) hucusque tractata etiam absque integratione perfici possint, brevitatis studio supprimitur.

Demonstrari potest, *omnis expressionis literam i continentis* (adeoque *hypothesi*, quod *detur i*, innixæ) *limitem, crescente i in infinitum, exprimere quantitatem plane pro Σ* (adeoque pro hypothesi *nullius i*), *siquidem non eveniant aequationes identicae.* Cave vero intelligas putari, *systema ipsum variari* posse (quod omnino *in se et per se determinatum* est) sed tantum *hypothesin*, quod *successive* fieri potest, donec non ad absurdum perducti fuerimus. *Posito* igitur, quod in *tali* expressione litera i pro casu, si S esset reipsa, *illam* quantitatem unicam designet, cuius $I = e$ sit; si vero *revera* Σ fuerit, *limes dictus* loco expressionis accipi *cogitetur*: manifesto *omnes* expressiones ex *hypothesi realitatis* ipsius S oriundæ (hoc sensu) *absolute valent*, etsi *prorsus ignotum sit, num Σ sit, aut non sit.*

Ita e. g. ex expressione in §. 30. obtenta facile (et quidem *tam* differentiationis auxilio, quam *absque* eo) valor notus pro Σ prodit

$$\bigcirc x = 2\pi x;$$

ex I. (§. 31.) rite tractato, sequitur

$$1 : \sin.\alpha = c : a;$$

ex II. vero

$$\frac{\cos.\alpha}{\sin.\beta} = 1, \quad \text{adeoque} \quad \alpha + \beta = R;$$

æquatio *prima* in III. fit identica, adeoque *valet* pro Σ, quamvis nihil in eo *determinet*; ex *secunda* autem fluit

$$c^2 = a^2 + b^2.$$

Aequationes notae fundamentales trigonometriae planae in Σ.

Porro inveniuntur (ex §. 32.) pro Σ area et corpus in III., utrumque

$$= pq;$$

ex IV.

$$\odot x = \pi x^2;$$

ex VII. sphæra radii x

$$=\frac{4}{3}\pi x^3$$

&c.

Sunt quoque theoremata ad finem (VI.) enuntiata manifesto *incondi-tionate vera.*

§. 33.

Superest adhuc, quid theoria ista sibi velit, (in §. 32. promissum) exponere.

I. Num Σ aut S aliquod *reipsa* sit, indecisum manet.

II. Omnia ex hypothesi *falsitatis* Ax. XI. deducta (semper *sensu* §. 32. intelligendo) *absolute* valent, adeoque *hoc sensu nulli hypothesi innituntur.* Habetur idcirco *trigonometria plana a priori*, in qua *solum* systema *verum ignotum* adeoque solummodo *absolutae* magnitudines expressionum incognitæ manent, per *unicum* vero casum notum, manifesto totum systema figeretur. Trigonometria sphærica autem in §. 26. absolute stabilitur. (Habeturque Geometria, Geometriæ planæ in Σ prorsus analoga in F).

III. Si *constaret* Σ esse, nihil hoc respectu amplius incognitum esset; si vero *constaret non esse* Σ, tunc (§. 31.) (e. g.) e lateribus x, y et angulo rectilineo ab iis intercepto, in *concreto datis* manifesto in se et per se impossibile esset triangulum absolute resolvere (i. e.) a priori determinare angulos ceteros et *rationem lateris tertii* ad duo data; nisi X, Y determinentur, ad quod *in concreto* haberi aliquod a oporteret, cuius A notum esset; atque tum i *unitas naturalis longitudinum* esset, (sicuti e est basis logarithmorum naturalium). Si existentia huius i constiterit; quomodo ad usum saltem quam exactissime construi possit, ostendetur.

IV. Sensu in I. et II. exposito patet, omnia in spatio methodo recentiorum Analytica (intra iustos fines valde laudanda) absolvi posse.

V. Denique lectoribus benevolis haud ingratum futurum est; pro casu illo, quodsi non Σ sed S reipsa esset, circulo æquale rectilineum construi.

§. 34.

Ex d *ducitur* dm ∥ an *modo sequente.* Fig. 12.

Fiat ex d

$$\mathfrak{db} \perp \mathfrak{an};$$

erigatur e puncto quovis aliquo a rectæ $\overline{\mathfrak{ab}}$

$$\mathfrak{ac} \perp \mathfrak{an} \text{ (in } \mathfrak{dba}),$$

et demittatur

$$\mathfrak{de} \perp \mathfrak{ac};$$

erit (§. 27.)

$$\bigcirc \mathfrak{ed} : \bigcirc \mathfrak{ab} = 1 : \sin. z,$$

siquidem *fuerit* dm ∥ bn.

Est vero sin. z non >1, adeoque ab non $>$ de. Descriptus igitur quadrans radio ipsi de æquali ex a in bac, gaudebit puncto aliquo b vel o cum $\mathfrak{b}\tilde{\mathfrak{d}}$ communi. Priore in casu manifesto $z = R$; in posteriore vero erit (§. 25.)

$$(\bigcirc \mathfrak{ao} = \bigcirc \mathfrak{ed}) : \bigcirc \mathfrak{ab} = 1 : \sin. \mathfrak{aob},$$

adeoque

$$z = \mathfrak{aob}.$$

Si itaque fiat $z = \mathfrak{aob}$, erit dm ∥ bn.

§. 35.

Si fuerit S reipsa; *ducetur recta ad anguli acuti crus unum perpendicularis, quae ad alterum* ∥ *sit, hoc modo.* Fig. 18.

Sit am ⊥ bc, et accipiatur ab = ac tam parvum (per §. 19.), ut si ducatur bn ∥ am (§. 34.), sit abn > angulo dato. Ducatur porro cp ∥ am (§. 34.), fiantque nbq, pcd utrumque æquale angulo dato; et $\mathfrak{b}\tilde{\mathfrak{q}}$, $\mathfrak{c}\tilde{\mathfrak{d}}$ se mutuo secabunt. Secet enim $\mathfrak{b}\tilde{\mathfrak{q}}$, (quod *per constr.* in nbc cadit) ipsam $\mathfrak{c}\tilde{\mathfrak{p}}$ in e; erit (propter bn ≃ cp) ebc < ecb, adeoque ec < eb. Sint

$$\mathfrak{ef} = \mathfrak{ec}, \quad \mathfrak{efr} = \mathfrak{ecd}, \quad \text{et} \quad \mathfrak{fs} \parallel \mathfrak{ep};$$

cadet fs in bfr. Nam cum bn ∥ cp, adeoque bn ∥ ep, atque bn ∥ fs sit;

erit (§. 14.)

$$\mathfrak{fbn} + \mathfrak{bfs} < 2R = \mathfrak{fbn} + \mathfrak{bfr};$$

itaque $\mathfrak{bfs} < \mathfrak{bfr}$. Quamobrem $\mathfrak{f}\overline{\mathfrak{r}}$ secat $\mathfrak{e}\overline{\mathfrak{p}}$, adeoque $\mathfrak{c}\overline{\mathfrak{d}}$ quoque ipsam $\mathfrak{e}\overline{\mathfrak{q}}$ in puncto aliquo $\mathfrak{d}$.

Sit iam $\mathfrak{dg} = \mathfrak{dc}$, atque $\mathfrak{dgt} = \mathfrak{dcp} = \mathfrak{gbn}$; erit (cum $\mathfrak{cd} \bumpeq \mathfrak{gd}$ sit)

$$\mathfrak{bn} \bumpeq \mathfrak{gt} \bumpeq \mathfrak{cp}.$$

Si fuerit lineæ *L* formis ipsius $\mathfrak{bn}$, punctum in $\mathfrak{b}\overline{\mathfrak{q}}$ cadens $\mathfrak{k}$ (§. 19.), et axis $\mathfrak{kl}$; erit

$$\mathfrak{bn} \bumpeq \mathfrak{kl},$$

adeoque

$$\mathfrak{bkl} = \mathfrak{bgt} = \mathfrak{dcp};$$

sed etiam

$$\mathfrak{kl} \bumpeq \mathfrak{cp}:$$

cadit ergo $\mathfrak{k}$ manifesto in $\mathfrak{g}$, estque $\mathfrak{gt} \,|||\, \mathfrak{bn}$. Si vero $\mathfrak{ho}$ ipsum $\mathfrak{bg}$ perpendiculariter bisecet; erit $\mathfrak{ho} \,|||\, \mathfrak{bn}$ constructum.

§. 36.

Fig. 10. Si fuerint data recta $\mathfrak{c}\overline{\mathfrak{p}}$ et planum $\overline{\mathfrak{mab}}$, atque fiat $\mathfrak{cb} \perp \overline{\mathfrak{mab}}$, (in $\overline{\mathfrak{bcp}}$) $\mathfrak{bn} \perp \mathfrak{bc}$, et $\mathfrak{cq} \,|||\, \mathfrak{bn}$ (§. 34.); *sectio ipsius* $\mathfrak{c}\overline{\mathfrak{p}}$ (si hæc in $\mathfrak{bcq}$ cadat) *cum* $\mathfrak{b}\overline{\mathfrak{n}}$ (in $\overline{\mathfrak{cbn}}$), adeoque *cum* $\overline{\mathfrak{mab}}$ reperitur. Et si fuerint data duo plana $\overline{\mathfrak{pcq}}$, $\overline{\mathfrak{mab}}$, et sit $\mathfrak{cb} \perp \overline{\mathfrak{mab}}$, $\mathfrak{cr} \perp \overline{\mathfrak{pcq}}$, atque (in $\overline{\mathfrak{bcr}}$) $\mathfrak{bn} \perp \mathfrak{bc}$, $\mathfrak{cs} \perp \mathfrak{cr}$; cadent $\mathfrak{bn}$ in $\overline{\mathfrak{mab}}$, et $\mathfrak{cs}$ in $\overline{\mathfrak{pcq}}$; et sectione ipsarum $\overline{\mathfrak{bn}}$, $\overline{\mathfrak{cs}}$ (si detur) reperta, erit perpendicularis in $\mathfrak{pcq}$ per eandem ad $\mathfrak{c}\overline{\mathfrak{s}}$ ducta manifesto *sectio ipsorum* $\overline{\mathfrak{mab}}$, $\overline{\mathfrak{pcq}}$.

§. 37.

Fig. 7. *In* $\overline{\mathfrak{am}} \,|||\, \mathfrak{bn}$ reperitur *tale* $\mathfrak{a}$, *ut sit* $\mathfrak{am} \bumpeq \mathfrak{bn}$; si (per §. 34.) construatur extra $\overline{\mathfrak{nbm}}$ $\mathfrak{gt} \,|||\, \mathfrak{bn}$, et fiant $\mathfrak{bg} \perp \mathfrak{gt}$, $\mathfrak{gc} = \mathfrak{gb}$, atque $\mathfrak{cp} \,|||\, \mathfrak{gt}$; ponaturque $\mathfrak{tg}\overline{\mathfrak{d}}$ ita, ut efficiat cum $\mathfrak{tg}\overline{\mathfrak{b}}$ angulum illi æqualem, quem $\mathfrak{pc}\overline{\mathfrak{a}}$ cum $\mathfrak{pc}\overline{\mathfrak{b}}$ facit; atque quæratur (per §. 36.) sectio $\overline{\mathfrak{dq}}$ ipsorum $\mathfrak{tg}\overline{\mathfrak{d}}$, $\mathfrak{nb}\overline{\mathfrak{a}}$; fiatque

$\mathfrak{ba} \perp \mathfrak{dq}$. Erit enimvero ob triangulorum *L* lineorum in *F* ipsius $\mathfrak{bn}$ exortorum similitudinem (§. 21.) manifesto $\mathfrak{db} = \mathfrak{da}$, et $\mathfrak{am} \simeq \mathfrak{bn}$.

Facile hinc patet (*L* lineis per *solos terminos* datis) reperiri posse etiam *terminos* proportionis quartum ac medium, atque omnes constructiones geometricas, quæ in Σ in plano fiunt, hoc modo in *F absque XI. Axiomate* perfici posse. Ita e. g. $4R$ in quotvis partes æquales geometrice dividi potest, si sectionem istam in Σ perficere licet.

§. 38.

Si construatur (per §. 37.) e. g. $\mathfrak{nbq} = \frac{1}{3} R$, et fiat (per §. 35.) in *S* ad $\mathfrak{b\bar{q}}$ perpendicularis $\mathfrak{am} \parallel \mathfrak{bn}$, atque determinetur (per §. 37.) $\mathfrak{jm} \simeq \mathfrak{bn}$; erit, si $\mathfrak{ja} = x$ sit, (§. 28.) Fig. 14.

$$X = 1 : \sin. \frac{1}{3} R = 2,$$

atque x *geometrice* constructum.

Et potest $\mathfrak{nbq}$ ita computari, ut $\mathfrak{ja}$ ab i quovis dato minus discrepet, cum nonnisi $\sin. \mathfrak{nbq} = \frac{1}{e}$ esse debeat.

§. 39

Si fuerint (in plano) $\mathfrak{pq}$ et $\mathfrak{st}$, $\parallel$ rectæ $\mathfrak{mn}$ (§. 27.), et $\mathfrak{ab}$, $\mathfrak{cd}$ sint perpendiculares ad $\mathfrak{mn}$ æquales; manifesto est Fig. 19.

$$\triangle \mathfrak{dec} \cong \triangle \mathfrak{bea},$$

adeoque anguli (forsan mixtilinei) $\mathfrak{ecp}$, $\mathfrak{eat}$ congruent, atque

$$\mathfrak{ec} = \mathfrak{ea}.$$

Si porro $\mathfrak{cf} = \mathfrak{ag}$, erit

$$\triangle \mathfrak{acf} \cong \triangle \mathfrak{cag},$$

et utrumque *quadrilateri* $\mathfrak{fagc}$ dimidium est. Si $\mathfrak{fagc}$, $\mathfrak{hagk}$ duo eiusmodi quadrilatera fuerint ad $\mathfrak{ag}$, inter $\mathfrak{pq}$ et $\mathfrak{st}$; æqualitas eorum (uti apud EUCLIDEM), nec non triangulorum $\mathfrak{agc}$, $\mathfrak{agh}$ eidem $\mathfrak{ag}$ insistentium,

verticesque in $\overline{\mathfrak{pq}}$ habentium, æqualitas patet. Est porro

$$\mathfrak{acf} = \mathfrak{cag},\ \mathfrak{gcq} = \mathfrak{cga},$$

atque

$$\mathfrak{acf} + \mathfrak{acg} + \mathfrak{gcq} = 2R$$

(§. 32.), adeoque etiam

$$\mathfrak{cag} + \mathfrak{acg} + \mathfrak{cga} = 2R;$$

itaque in quovis eiusmodi triangulo $\mathfrak{acg}$ summa trium angulorum $= 2R$.

Sive in $\mathfrak{ag}$ (quæ $\parallel \mathfrak{mn}$) ceciderit autem *recta* $\mathfrak{ag}$, sive non; triangulorum *rectilineorum* $\mathfrak{agc}$, $\mathfrak{agh}$ *tam ipsorum, quam summarum angulorum ipsorumdem, aequalitas* in aperto est.

§. 40.

Fig. 20. *Aequalia triangula* $\mathfrak{abc}$, $\mathfrak{abd}$ *(abhinc rectilinea) uno latere aequali gaudentia, summas angulorum aequales habent.*

Nam dividat $\mathfrak{mn}$ bifariam tam $\mathfrak{ac}$ quam $\mathfrak{bc}$, et sit $\mathfrak{pq}$ (per $\mathfrak{c}$) $\parallel \mathfrak{mn}$; cadet $\mathfrak{d}$ in $\overline{\mathfrak{pq}}$. Nam si $\overline{\mathfrak{bd}}$ ipsum $\overline{\mathfrak{mn}}$ in puncto $\mathfrak{e}$, adeoque (§. 39.) ipsum $\overline{\mathfrak{pq}}$ ad distantiam $\mathfrak{ef} = \mathfrak{eb}$ secet; erit

$$\triangle\,\mathfrak{abc} = \triangle\,\mathfrak{abf},$$

adeoque et

$$\triangle\,\mathfrak{abd} = \triangle\,\mathfrak{abf},$$

unde $\mathfrak{d}$ in $\mathfrak{f}$ cadit: si vero $\overline{\mathfrak{bd}}$ ipsum $\overline{\mathfrak{mn}}$ non secuerit, sit $\mathfrak{c}$ punctum, ubi perpendicularis rectam $\mathfrak{ab}$ bisecans ipsum $\overline{\mathfrak{pq}}$ secat, atque $\mathfrak{gs} = \mathfrak{ht}$ ita, ut $\overline{\mathfrak{st}}$ *productam* $\mathfrak{bd}$ in puncto aliquo $\mathfrak{k}$ secet (quod fieri posse modo simili patet, ut §. 4.); sint porro $\mathfrak{sl} = \mathfrak{sa}$, $\mathfrak{lo} \parallel \mathfrak{st}$, atque $\mathfrak{o}$ sectio ipsorum $\overline{\mathfrak{bk}}$ et $\overline{\mathfrak{lo}}$; esset tum (§. 39.)

$$\triangle\,\mathfrak{abl} = \triangle\,\mathfrak{abo},$$

adeoque

$$\triangle\,\mathfrak{abc} > \triangle\,\mathfrak{abd}$$

(contra hyp.).

§. 41.

Aequalia triangula abc, def *aequalibus angulorum summis gaudent.* Fig. 21.

Nam secet mn tam ac quam bc, ita pq tam df quam fe bifariam, et sit rs || mn, atque to || pq; erit perpendicularis ag ad rs aut æqualis perpendiculari dh ad to, aut altera e. g. dh erit maior: in quovis casu ○ df e centro a cum gs punctum aliquod k commune habet, eritque (§. 39.)

$$\triangle\, abk = \triangle\, abc = \triangle\, def.$$

Est vero △ akb (per §. 40.) triangulo dfe, ac (per §. 39.) triangulo abc *aequiangulum.* Sunt igitur etiam triangula abc, def æquiangula.

In *S converti* quoque theorema potest. Sint enim triangula abc, def reciproce æquiangula, atque △ bal = △ def; erit (per præc.) alterum alteri, adeoque etiam △ abc triangulo abl æquiangulum, et hinc manifesto

$$bcl + blc + cbl = 2R.$$

Atqui (ex §. 31.) cuiusvis trianguli angulorum summa in *S* est $< 2R$: cadit igitur l in c.

§. 42.

Si fuerit complementum summae angulorum trianguli abc *ad* $2R$ Fig. 22.

$$u,$$

trianguli def *vero*

$$v;$$

est

$$\triangle\, abc : \triangle\, def = u : v.$$

Nam si quodvis triangulorum acg, gch, hcb, dfk, kfe sit $= p$, atque

$$\triangle\, abc = mp, \quad \triangle\, def = np;$$

sitque s summa angulorum cuiusvis trianguli, quod $= p$ est: erit manifesto

$$2R - u = ms - (m - 1)\,2R = 2R - m\,(2R - s),$$

et

$$u = m\,(2R - s),$$

et pariter

$$v = n\,(2R - s).$$

Est igitur

$$\triangle\,\mathfrak{abc} : \triangle\,\mathfrak{def} = m : n = u : v.$$

Ad casum incommensurabilitatis triangulorum $\mathfrak{abc}$, $\mathfrak{def}$ quoque extendi facile patet.

Eodem modo demonstratur triangula in superficie sphærica esse uti *excessus* summarum angulorum eorundem supra $2R$. Si duo anguli trianguli sphærici recti fuerint, tertius z erit excessus dictus; est autem triangulum istud (peripheria maxima p dicta) manifesto

$$= \frac{z}{2\pi}\,\frac{p^2}{2\pi} \quad (\text{§. 32. VI.});$$

consequenter quodvis triangulum, cuius angulorum excessus $= z$, est

$$= \frac{zp^2}{4\pi^2}.$$

§. 43.

Fig. 15. Iam *area* trianguli rectilinei in S per summam angulorum exprimetur.

Si $\mathfrak{ab}$ crescat in infinitum; erit (§. 42.)

$$\triangle\,\mathfrak{abc} : (R - u - v)$$

constans. Est vero

$$\triangle\,\mathfrak{abc} \simeq \mathfrak{bacn} \quad (\text{§. 32. V.})$$

et

$$R - u - v \simeq z \quad (\text{§. 1.});$$

adeoque

$$\mathfrak{bacn} : z = \triangle\,\mathfrak{abc} : (R - u - v) = \mathfrak{bac'n'} : z'.$$

Est porro manifesto

$$\mathfrak{bdcn} : \mathfrak{bd'c'n'} = r : r' = \text{tang.}\,z : \text{tang.}\,z' \quad (\text{§. 30.}).$$

Pro $y' \backsim 0$ autem est

$$\frac{\mathfrak{bd'c'n'}}{\mathfrak{bac'n'}} \backsim 1,$$

nec non

$$\frac{\text{tang.}\, z'}{z'} \backsim 1;$$

consequ.

$$\mathfrak{bdcn} : \mathfrak{bacn} = \text{tang.}\, z : z.$$

Erat vero (§. 32.)

$$\mathfrak{bdcn} = ri = i^2 \,\text{tang.}\, z;$$

est igitur

$$\mathfrak{bacn} = zi^2.$$

Quovis triangulo, cuius angulorum summæ complementum ad $2R$ z est, in posterum breviter $\triangle$ dicto, erit idcirco

$$\triangle = zi^2.$$

Facile hinc liquet, quod si Fig. 14.

$$\mathfrak{or} \parallel \mathfrak{am} \quad \text{et} \quad \mathfrak{ro} \parallel \mathfrak{ab}$$

fuerint; *area* inter $\overline{\mathfrak{or}}$, $\overline{\mathfrak{st}}$, $\overline{\mathfrak{bc}}$ comprehensa (quæ manifesto limes absolutus est areæ triangulorum rectilineorum sine fine crescentium, seu ipsius $\triangle$ pro $z \backsim 2R$), sit

$$= \pi i^2 = \odot i \text{ in } F.$$

Limite isto per $\square$ denotato, erit porro (per §. 30.) Fig. 15.

$$\pi r^2 = \text{tang.}\, z^2 \square = \odot r \text{ in } F \text{ (§. 21.)}$$
$$= \odot s \text{ (per §. 32. VI.)},$$

si chorda $\mathfrak{dc}$ s dicatur. Si iam radio dato s, circuli in plano (sive radio L formi circuli in F) perpendiculariter bisecto, construatur (per §. 34.) $\mathfrak{db} \parallel\!\!\!\simeq \mathfrak{cn}$; demissa perpendiculari $\mathfrak{ca}$ ad $\mathfrak{db}$, et erecta perpendiculari $\mathfrak{cm}$ ad $\mathfrak{ca}$; habebitur z; unde (per §. 37.) tang. z^2, radio L formi ad lubitum pro unitate assumto, *geometrice determinari potest per duas lineas uniformes eiusdem curvaturae* (quæ solis terminis datis, constructis axibus, manifesto tanquam rectæ commensurari, atque hoc respectu rectis æquivalentes spectari possunt).

Fig. 23 Porro construitur quadrilaterum ex. gr. regulare $= \square$, ut sequitur. Sit

$$\mathfrak{abc} = R,\ \mathfrak{bac} = \frac{1}{2} R,\ \mathfrak{acb} = \frac{1}{4} R,\ \text{et}\ \mathfrak{bc} = x;$$

poterit X (ex §. 31. II.) per meras radices quadraticas exprimi, et (per §. 37.) construi: habitoque X, (per §. 38., sive etiam 29. et 35.) x ipsum determinari potest. Estque octuplum $\triangle \mathfrak{abc}$ manifesto $= \square$, atque *per hoc, circulus planus radii s, per figuram rectilineam, et lineas uniformes eiusdem generis (rectis, quoad comparationem inter se, aequivalentes) geometrice quadratus*; *circulus F formis vero eodem modo complanatus*: *habeturque aut Axioma XI. Euclidis verum, aut quadratura circuli geometrica*; etsi hucusque indecisum manserit, quodnam ex his duobus revera locum habeat. Quoties tang. z^2 *vel* numerus integer *vel* fractio rationalis fuerit, cuius (ad simplicissimam formam reductæ) denominator *aut* numerus primus formæ $2^m + 1$ (cuius est etiam $2 = 2^0 + 1$) *aut* productum fuerit e quotcunque primis huius formæ, quorum (ipsum 2, qui solus quotvis vicibus occurrere potest, excipiendo) quivis *semel* ut factor occurrit: per theoriam polygonorum ill. *GAUSS* (præclarum nostri imo omnis ævi inventum), etiam ipsi tang. $z^2 \square = \odot s$ (et nonnisi pro talibus valoribus ipsius z) figuram rectilineam æqualem constituere licet. Nam *divisio* ipsius $\square$ (theoremate §. 42. facile ad quælibet polygona extenso) manifesto *sectionem* ipsius $2R$ requirit, quam (ut ostendi potest) unice sub dicta conditione geometrice perficere licet. In omnibus autem talibus casibus præcedentia facile ad scopum perducent. Et potest quævis figura rectilinea in polygonum regulare n laterum geometrice converti, siquidem n sub formam *GAUSSianam* cadat.

Superesset denique, (ut res omni numero absolvatur), impossibilitatem (absque suppositione aliqua) decidendi, num Σ aut aliquod (et quodnam) S sit, demonstrare: quod tamen occasioni magis idoneæ reservatur.

WOLFGANGI BOLYAI
ADDITAMENTUM AD APPENDICEM.

Denique *aliquid Auctori Appendicis proprium*, *coronidis instar*, addere fas sit: qui tamen ignoscat, si quid non acu eius tetigerim.

Res breviter in eo consistit: *formulae trigonometriae sphaericae*, in Appendice dicta ab axiomate XI. Eucl. independenter demonstratæ, *cum formulis trigonometriae planae conveniunt, si* (modo statim dicendo) *latera trianguli sphaerici realia*, *rectilinei vero imaginaria accipiantur*; adeo ut quoad formulas trigonometricas planum ut sphæra imaginaria considerari possit, si pro reali illa accipiatur, in qua sin. $R=1$.

Pro casu, si axioma Eucl. verum non fuerit, demonstratur (Appendix §. 30.) dari certum i, pro quo ibidem dictum I est $=e$ (basi logarithmorum naturalium), atque pro hoc casu formulæ trigonometriæ planæ quoque demonstrantur (ibidem §. 31.); et quidem ita, ut (iuxta §. 32., post VII., ibidem) formulæ et pro casu veritatis axiomatis dicti valeant; nempe si supponendo, quod $i \to \infty$, limites valorum accipiantur; nimirum systema Euclideum est quasi limes systematis antieuclidei (pro $i \to \infty$). Ponatur, pro casu existentis i, unitas $=i$, atque conceptus *sinus cosinusque* extendatur et ad arcus imaginarios; ita ut arcum sive realem sive imaginarium denotet p, dicatur

$$\frac{1}{2}(e^{p\sqrt{-1}}+e^{-p\sqrt{-1}})$$

cosinus ipsius p, et

$$\frac{1}{2\sqrt{-1}}(e^{p\sqrt{-1}}-e^{-p\sqrt{-1}})$$

dicatur sinus ipsius p.

Erit hinc pro q reali

$$\frac{1}{2\sqrt{-1}}(e^q - e^{-q}) = \frac{1}{2\sqrt{-1}}(e^{-q\sqrt{-1}\sqrt{-1}} - e^{q\sqrt{-1}\sqrt{-1}}) =$$

$$= \sin.(-q\sqrt{-1}) = -\sin. q\sqrt{-1}$$

Ita

$$\frac{1}{2}(e^q + e^{-q}) = \frac{1}{2}(e^{-q\sqrt{-1}\sqrt{-1}} + e^{q\sqrt{-1}\sqrt{-1}}) =$$

$$= \cos.(-q\sqrt{-1}) = \cos. q\sqrt{-1};$$

si nempe et in circulo imaginario sinus negativi arcus sinui arcus positivi alioquin priori æqualis sit, præterquam quod negativus sit, atque cosinus arcus positivi et negativi (si alioquin æquales fuerint), sit idem.

In Appendice dicta §. 25. demonstratur absolute, id est ab axiomate dicto independenter; quod in quovis triangulo rectilineo *sinus angulorum sint, uti peripheriae radiorum lateribus oppositis aequalium*; demonstraturque porro, pro casu existentis i, peripheriam radii y esse

$$= \pi i (e^{\frac{y}{i}} - e^{-\frac{y}{i}}),$$

quod pro $i = 1$ fit

$$\pi(e^y - e^{-y}).$$

Itaque (§. 31. ibidem) pro triangulo rectilineo rectangulo, cuius catheti sunt a et b, hypotenusa c, et anguli lateribus a, b, c oppositi sunt α, β, π; est (pro ($i = 1$)

in I.

$$1 : \sin. \alpha = \pi(e^c - e^{-c}) : \pi(e^a - e^{-a});$$

adeoque

$$1 : \sin. \alpha = \frac{1}{2\sqrt{-1}}(e^c - e^{-c}) : \frac{1}{2\sqrt{-1}}(e^a - e^{-a}).$$

Unde

$$1 : \sin. \alpha = -\sin. c\sqrt{-1} : -\sin. a\sqrt{-1}.$$

Et hinc

$$1 : \sin. \alpha = \sin. c\sqrt{-1} : \sin. a\sqrt{-1}.$$

In II. fit

$$\cos.\alpha : \sin.\beta = \cos.a\sqrt{-1} : 1.$$

In III. fit

$$\cos.c\sqrt{-1} = \cos.a\sqrt{-1}\,\cos.b\sqrt{-1}.$$

Quæ prouti omnes exinde promanantes formulæ trigonometriæ planæ, cum formulis trigonometriæ sphæricæ prorsus conveniunt; nisi quod si ex. gr. trianguli sphærici rectanguli quoque catheti angulique iis oppositi, hypotenusaque nomina eadem sortiantur, latera trianguli rectilinei per $\sqrt{-1}$ dividenda sint, ut formulæ pro sphæricis prodeant.

Nempe ex I. fiet

$$1 : \sin.\alpha = \sin.c : \sin.a,$$

ex II. fiet

$$1 : \cos.a = \sin.\beta : \cos.\alpha,$$

ex III. fiet

$$\cos.c = \cos.a\,\cos.b.$$

Quum ceteris supersedere liceat, et lectorem deductione (App. §. 32. post VII.) omissa offendi impedirique expertus sim: haud abs re erit ostendere, quomodo ex. gr. ex

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \frac{1}{2}\left(e^{\frac{a}{i}} + e^{-\frac{a}{i}}\right)\left(e^{\frac{b}{i}} + e^{-\frac{b}{i}}\right)$$

sequatur

$$c^2 = a^2 + b^2$$

(theorema Pythagoreum pro systemate Euclideo); verosimiliter Auctor quoque ita deduxit, et ceteræ quoque eodem modo sequuntur.

Est nempe potentiis ipsius e per series expressis

$$e^{\frac{k}{i}} = 1 + \frac{k}{i} + \frac{k^2}{2i^2} + \frac{k^3}{2.3i^3} + \frac{k^4}{2.3.4i^4} + \cdots$$

$$e^{-\frac{k}{i}} = 1 - \frac{k}{i} + \frac{k^2}{2i^2} - \frac{k^3}{2.3i^3} + \frac{k^4}{2.3.4i^4} - \cdots;$$

adeoque

$$e^{\frac{k}{i}} + e^{-\frac{k}{i}} = 2 + \frac{k^2}{i^2} + \frac{k^4}{3.4i^4} + \cdots$$

$$= 2 + \frac{k^2 + u}{i^2},$$

(si omnium terminorum post $\frac{k^2}{i^2}$ summa $\frac{u}{i^2}$ dicatur); estque $u \sim 0$, dum $i \sim \infty$. Nam multiplicentur omnes termini post $\frac{k^2}{i^2}$ per i^2; erit terminus primus $\frac{k^4}{3 \cdot 4 i^2}$, et quivis exponens $< \frac{k^2}{i^2}$; essetque etsi exponens ubique hic maneret, summa

$$\frac{k^4}{3 \cdot 4 i^2} : \left(1 - \frac{k^2}{i^2}\right) = \frac{k^4}{3 \cdot 4 (i^2 - k^2)},$$

quod manifesto ~ 0, dum $i \sim \infty$.

Atque ex

$$e^{\frac{c}{i}} + e^{-\frac{c}{i}} = \frac{1}{2}\left(e^{\frac{a+b}{i}} + e^{-\frac{a+b}{i}} + e^{\frac{a-b}{i}} + e^{-\frac{a-b}{i}}\right)$$

sequitur (pro ω, v, λ adinstar u acceptis)

$$2 + \frac{c^2 + \omega}{i^2} = 1 + \frac{(a+b)^2 + v}{2i^2} + 1 + \frac{(a-b)^2 + \lambda}{2i^2}.$$

Atque hinc

$$c^2 = \frac{1}{2}(a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + v + \lambda - 2\omega),$$

quod

$$\sim a^2 + b^2.$$

Scholion. Sphæræ illius, in qua sinus totus est $1 = i$, radius est ordinata y lineæ L formis ipsi $i = 1$ æqualis, ad axem per unam extremitatem ex altera perpendiculariter missa. Nempe *in superficie* (App. §. 21.) *F dicta, tota Geometria Euclidea valet, lineis L vicem rectarum subeuntibus*: atque pro radio L formi $= 1$, qui sinus totus in F erit, peripheriæ eiusdem radius in plano erit plane dictum y; quod ad sphæram imaginariam, ad quam planum (in systemate antieuclideo) revocatur, facile applicatur.

ADNOTATIONES EDITORUM.

In Ed. I. Appendicis literæ singulares puncta denotantes literis quæ dicuntur *cursivis* impressæ sunt, sed in libellis nobis relictis manu Ioannis Bolyai scriptis puncta literis qu. d. *fractur* denotantur, quibus etiam pater eius usus est.

Pag. 1. post ỷ. 15. in Ed. I. legitur:

$\llcorner$ denotet perpendiculare
$\wedge$ « angulum.

Nos hæc delevimus, quia pro signo $\llcorner$ usitatiore $\perp$ utimur, vocabulum «angulum» autem ubique integrum scribimus.

Pag. 8. ỷ. ultimo «$\mathfrak{krs}$» correximus in «$\mathfrak{hrs}$», ut et ipse auctor correxit in libello manu scripto, quo Appendicem lingua Germanica denuo pertractavit.

Pag. 10. ỷ. ultimo §. 18. «$\bigcirc\,\mathfrak{ha}$» emendavimus ex «$\bigcirc\,\mathfrak{ht}$».

Pag. 14. extremae §. 26. auctor ascripsit exemplari in bibliotheca Academiæ Hungaricæ asservato «cos. quoque necessarium». Videtur in editione quadam altera propositionem de cosinu demonstraturus fuisse, quamquam nihil hic maioris momenti vere desideratur. Revera considerationes, quibus in «Supplemento numeri ·31351» Tom. II. Tentaminis ex

$$\sin. H : \sin. A = 1 : \sin. a$$

trigonometria sphærica integra deducitur, et in geometria absoluta valent.

Pag. 16. ỹ. 3. Ex libello Germanico auctoris manu scripto «$\mathfrak{cq}$» scripsimus pro vitioso «$\mathfrak{cg}$» Editionis I.

Pag. 17. ỹ. 5. «$\mathfrak{cn} \parallel\!| \mathfrak{ab}$, $\mathfrak{c'n'} \parallel\!| \mathfrak{ab}$» scripsimus pro «$\mathfrak{cn}$, $\mathfrak{c'n'} \parallel\!| \mathfrak{ab}$» Ed. I.

Pag. 20. ỹ. 8. «et 27» Ioannes Bolyai ipse ascripsit in exemplari Academiæ.

Pag. 22. ỹ. 4. a calce «§. 30.». scripsimus pro «§. 29.». Ed. I.

Pag. 35. In hoc Additamento, quod pagg. 380—383 Tom. II. Ed. I. Tentaminis continetur, delevimus mentionem de paginis huius operis factam, in quibus solum theoremata omnibus nota continentur. Articulum vero secundum in hunc locum reiecimus:

> «Pro v positivo radicem positivam ipsius $-v^2$ per $\dotplus v$ denotat, et negativam per $\dot{-}v$, sed ob defectum signorum (quum vix hæc duo quadamtenus prodierunt), radix positiva ipsius -1 per $\sqrt{-1}$ et negativa per $-\sqrt{-1}$ denotabitur».

Ibidem ỹ. 3. Post «Appendicis» delevimus «in tomo primo».

Ibidem ỹ. 11. Initium articuli huius in Ed. I. sic legitur:

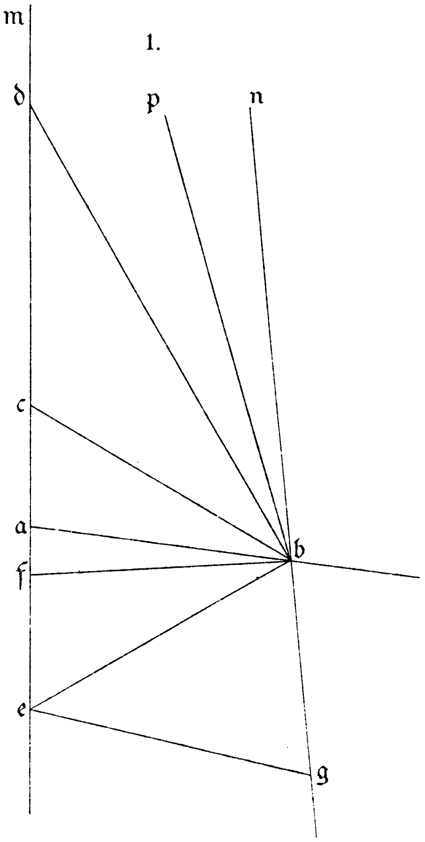
> «Nimirum de axiomate Euclideo dictum in tomo primo satis superque est: pro casu, si verum non fuerit &».

Pag. 38. ỹ. 2. «multiplicentur» scripsimus pro erroneo «dividantur».

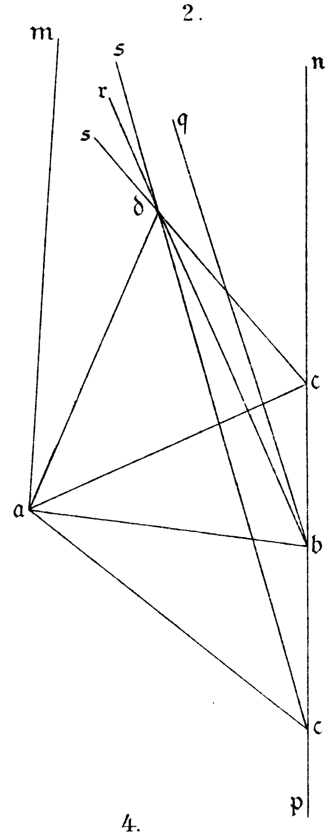
Ibidem ỹ. 8. Latus sinistrum formulæ in Ed. I. deest.

Tab. I. Fig. 1-4.

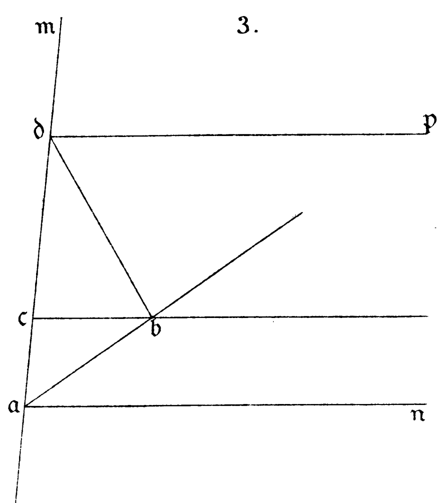
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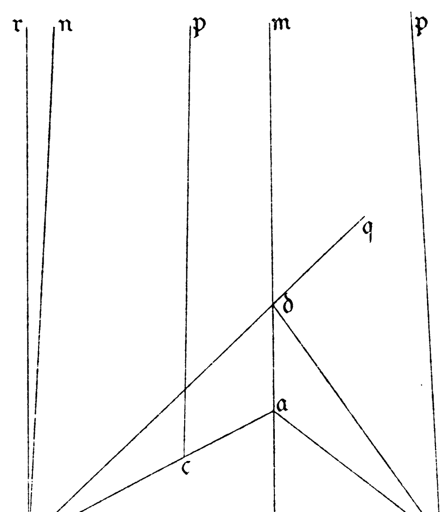
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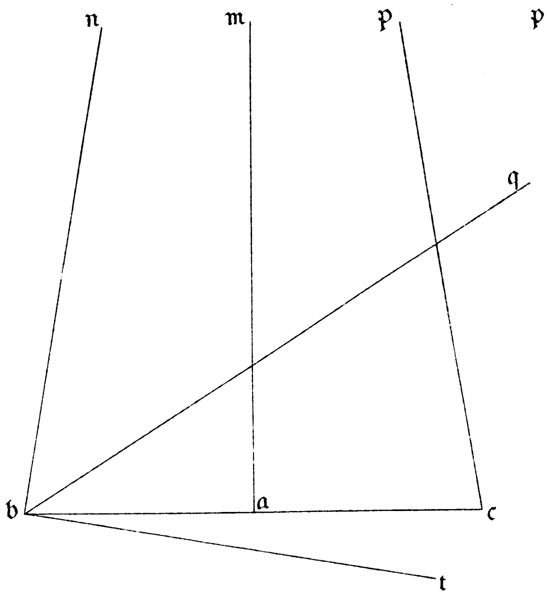


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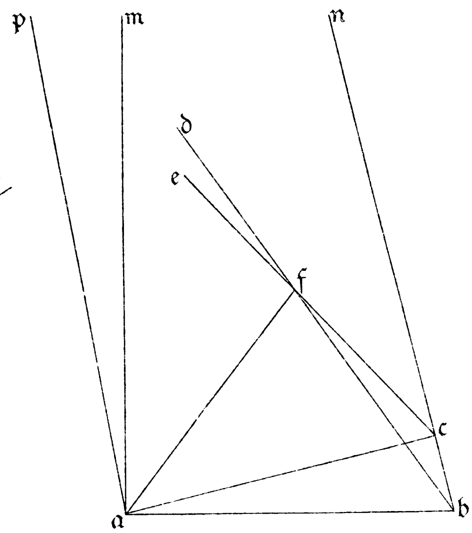


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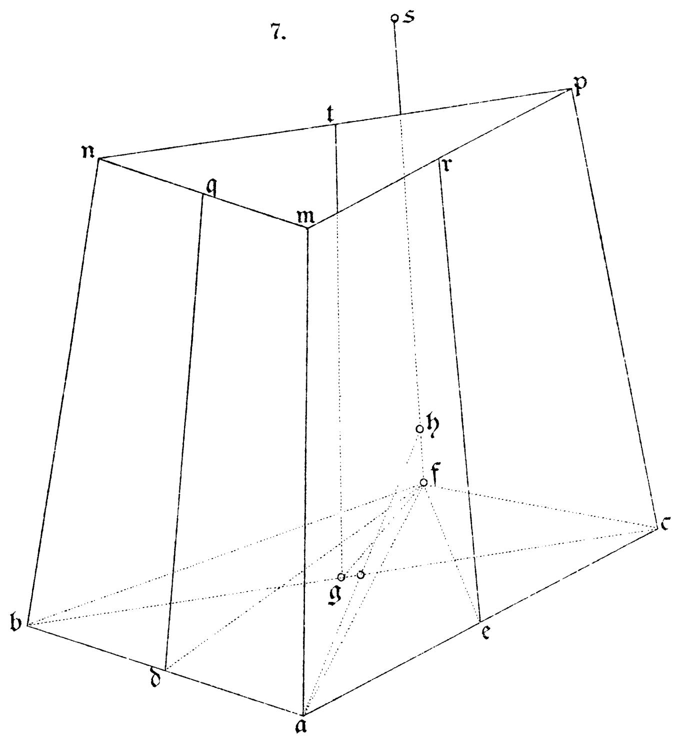
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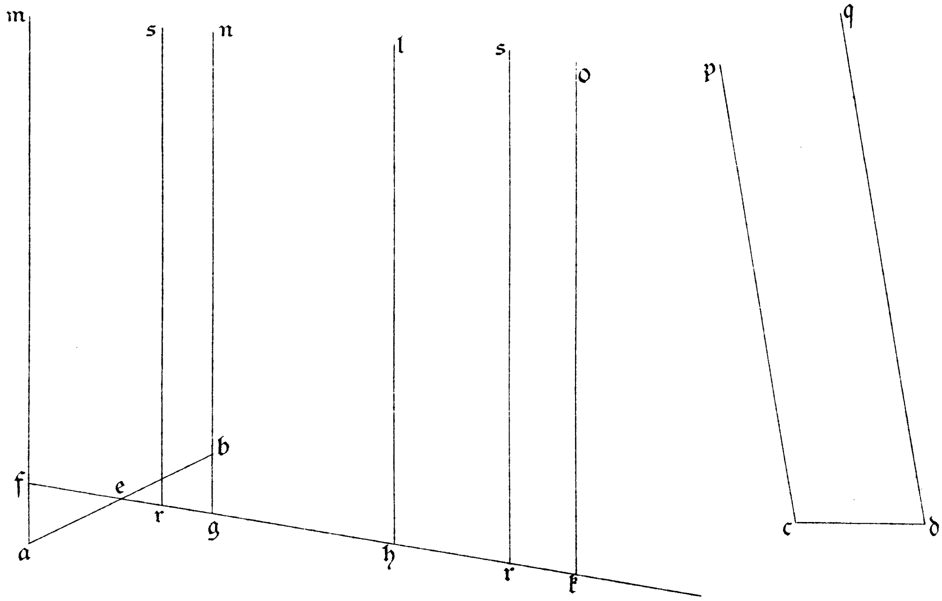


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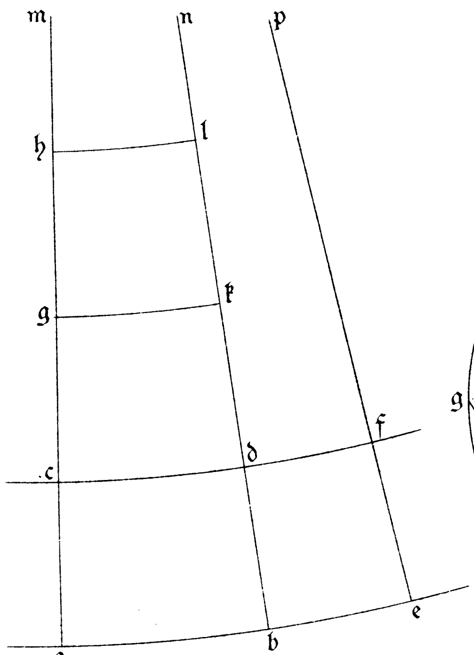


Tab. II. Fig. 5-7.

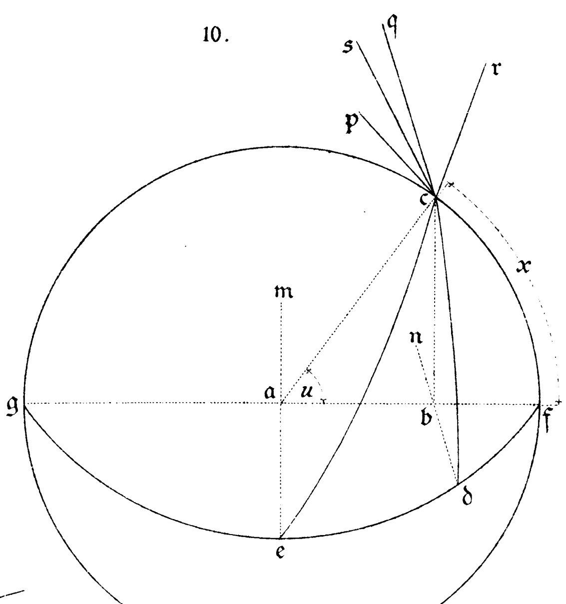
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9.



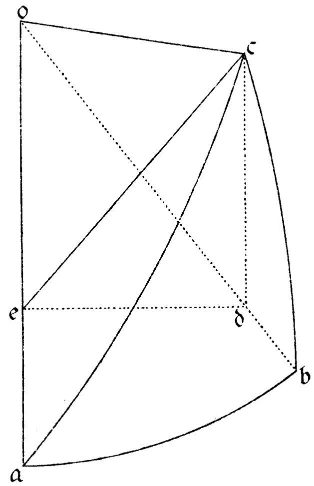
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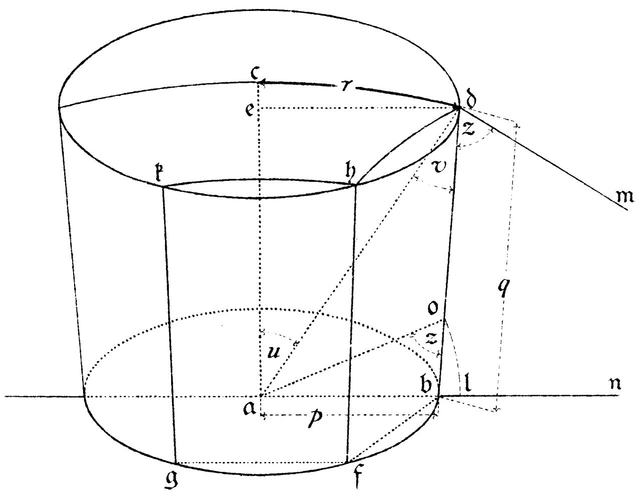
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Tab. III. Fig. 8.–10.

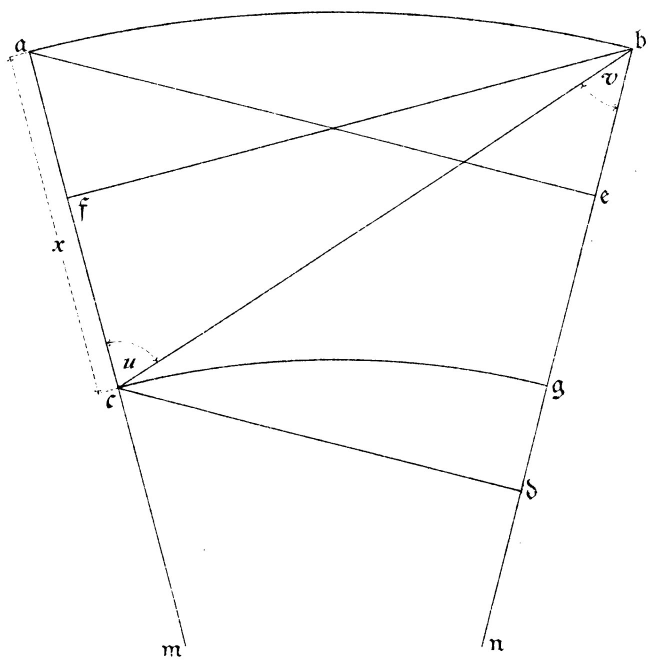
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12.



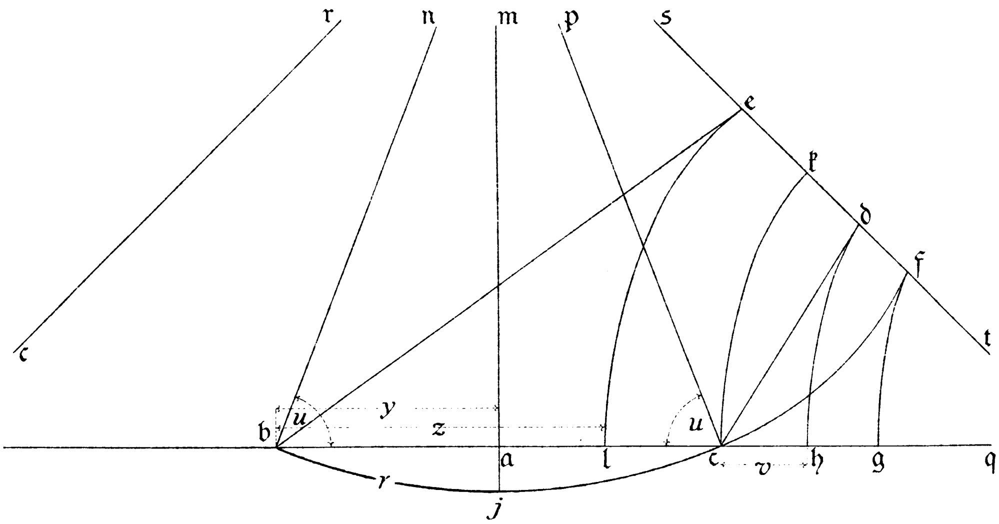
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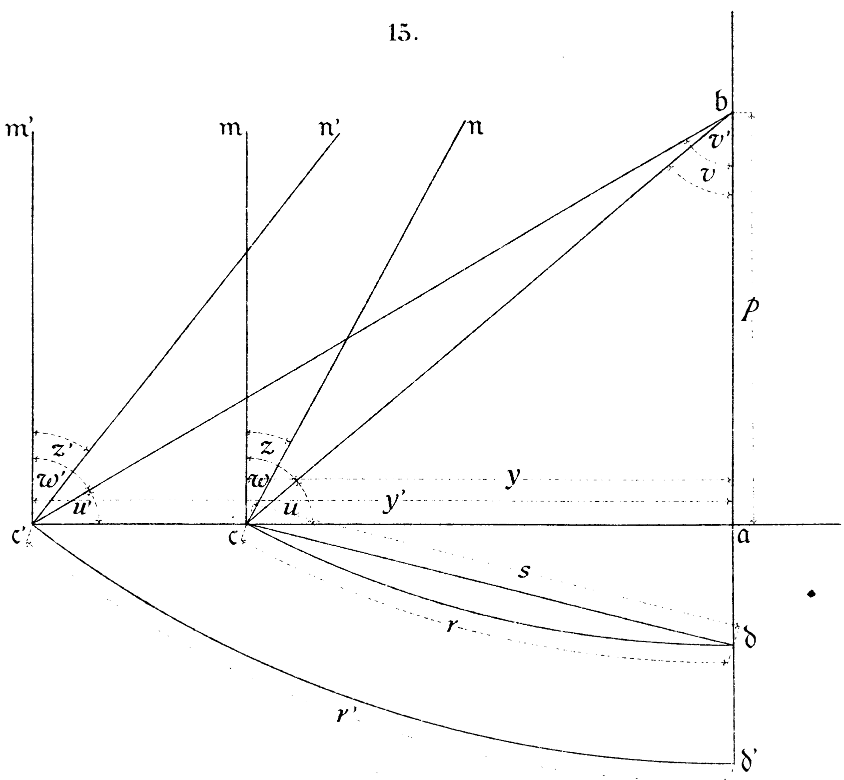
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Tab. IV. Fig. 11-13.

14.



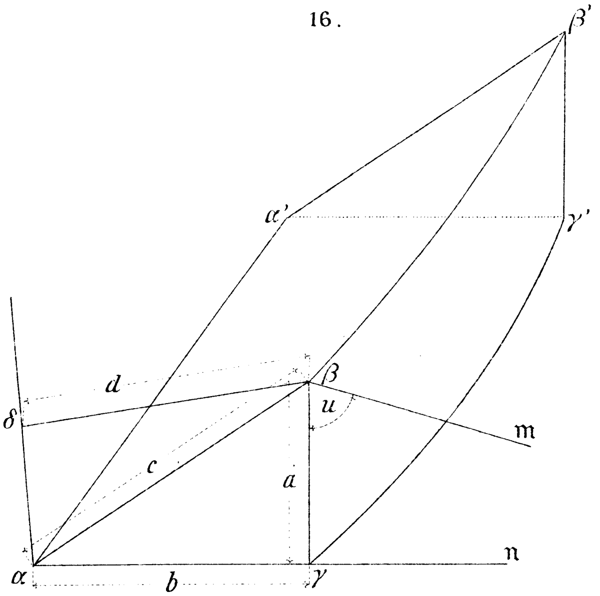
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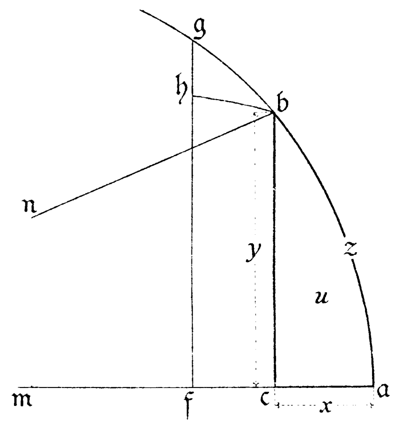
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Tab. V. Fig. 14–15.

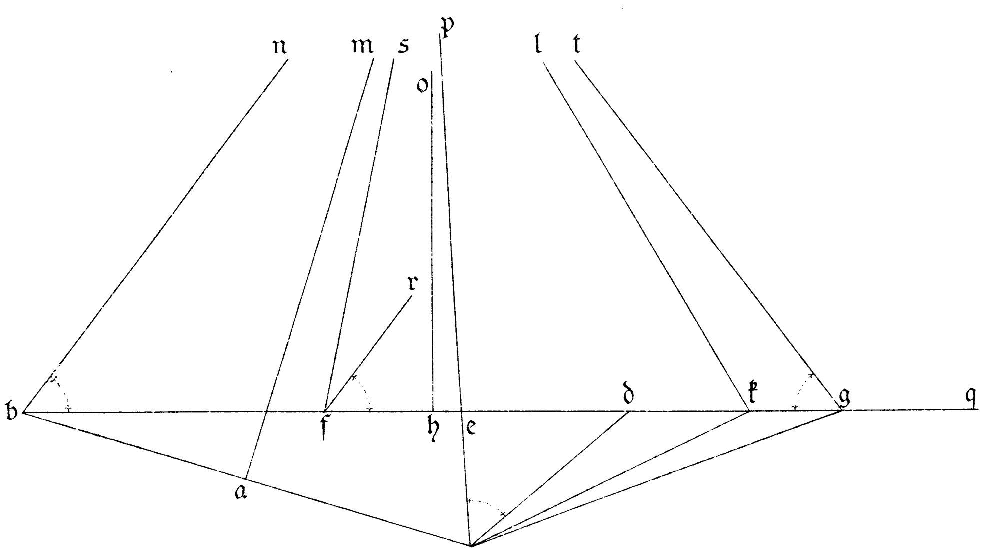
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17.



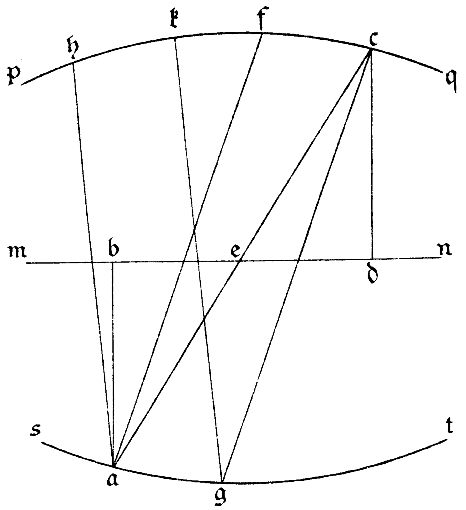
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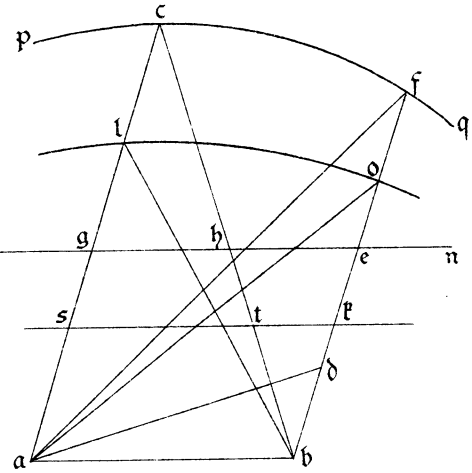
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Tab. VI. Fig. 16 – 18.

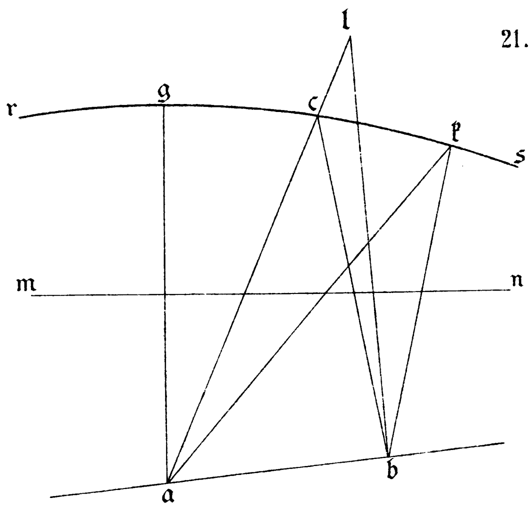
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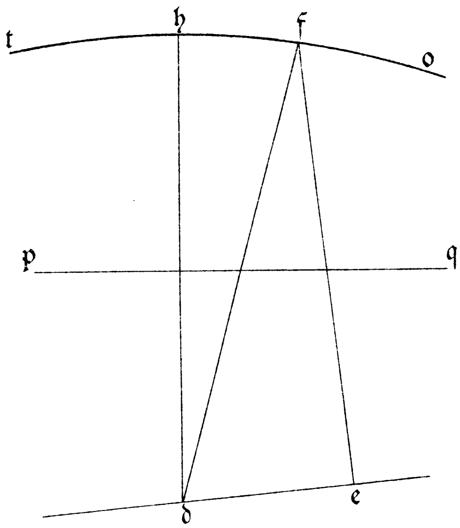


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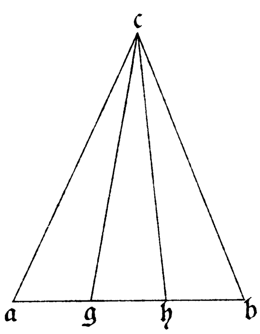


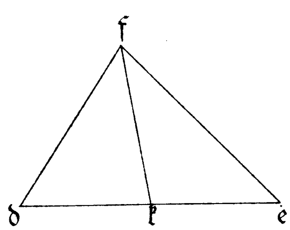
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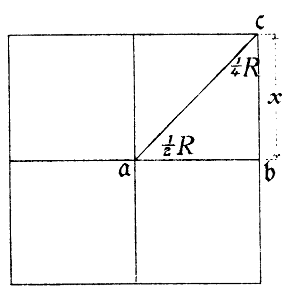


22.





23.



Del. TŐTÖSSY B.

Lith. GRUND V. utódai.

Tab. VII. Fig. 19.–23.